A CHEBYSHEV COLLOCATION BLOCK METHOD FOR SOLUTION OF THIRD ORDER INITIAL VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

by

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Abstract

This work focuses on formulation of a continuous scheme for the numerical solution of third order Initial value problems in ordinary differential equations. The collocation approach is adopted and Chebyshev is employed as basis function. An analysis of the method shows that the proposed method is zero-stable, consistent and hence convergent. On comparison, the method performed favourably compared with the existing ones.

INTRODUCTION

The general third order Initial Value Problems (IVPs) of Ordinary Differential Equations (ODEs) of the form $y''' = f(x, y, y', y'')$, $y(a) = \alpha$, $y'(a) = \beta$, $y''(a) = \gamma$, $x \in (a,b)$ (1)

Where f is continuous in [a, b] has been practically used in a wide variety of applications, especially in science and engineering field such as satellite tracking/warning systems, celestial mechanics, mass action kinetics, solar systems and molecular biology (Aladeselu 2007).Many of such problems do not have analytical solution hence the development of numerical schemes to obtain approximate solution of (1) becomes necessary. Various numerical schemes for solving differential equations exist in literature. Among these are the Runge-kutta, Taylor's algorithm and the Linear Multistep Methods. Presently, Linear Multistep Methods (LMMs) are very popular and useful among these methods. They are suitable in providing solutions to ODEs within a given interval and they are useful for information about the solution at more than one point. Problems arising from ODEs can either be formulated as an IVP or a BVP. However, our concern shall be with IVP. Several researchers such as Lambert (1991), Adesanya et al (2009), Olabode and Yusuph (2009) attempted the solution of (1) using LMMs without reduction to system of first order ODEs. Olabode and Yusuph (2009), Adeyefa (2017) and Kuboye et al. (2018) developed new block methods which possess the desirable feature of Runge-Kutta method of being self-starting and eliminated the use of predictor using power series as basis function. However, due to the elegant properties of Chebyshev polynomials, we shall be focusing on the development of new block methods with the use of Chebyshev polynomials as basis function.

2. BLOCK METHODS

Block methods are formulated in terms of LMMs. They provide the traditional advantage of one-step methods e.g., Runge-Kutta methods, of being self-starting and permitting easy change of step length (Lambert, 1973). Another important feature of the block approach is that all the discrete schemes are of uniform order and are obtained from a single continuous formula in contrast to the non-self-starting predictor-corrector approach. This self-starting method was used by Anake (2013) to derive a class of one-step hybrid methods for the numerical solution of second order ordinary differential equation with power series as the basis function. Lately, Adeyefa (2014) adopted this same approach but employed Chebyshev

Polynomial to develop a set of algorithms. The numerical solutions obtained by these researchers are desirable as their methods at many points recovered the exact solutions. In what follows, we shall adopt the block method approach to formulate a third order numerical scheme using Chebyshev polynomial as our basis function.

3. **DEVELOPMENT OF THE METHOD**

A continuous representation of a two-step hybrid method which will be used to generate the main method and other methods required to set up the block methods shall be derived in this section. Here, we set out by approximating the analytical solution of problem (1) with a polynomial of the form

$$y(x) = \sum_{n=0}^{r+s-1} a_n T_n(x) \tag{2}$$

on the partition $a = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_k = b$ of the integration interval [a, b], with a constant step size h, given by $h = x_{k+1} - x_k$; $k = 0,1,\ldots K-1$ as basis or trial function.

The polynomial $T_n(x)$ in (2) is defined by

$$T_n(x) = Cos(nCos^{-1}x) \equiv \sum_{j=0}^{n} C_j^{(n)} x^n \tag{3}$$

and it is the nth degree Chebyshev polynomial which is valid in the range of definition of (2). The Chebyshev polynomials $T_n(x)$ satisfied the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),\ x \in [-1,1],\ n \geq 1\ (T_0(x) = 1) \tag{4}$$

Thus, we deduced that

$$T_1(x) = X = \frac{2x - x_{k+1} - x_k}{x_{k+1} - x_k} = \frac{2x - 2x_k - h}{h} = t\ ,\ x_k \leq x \leq x_{k+1} \tag{5}$$

where $t = t(x)$,a function of x , is given by (5).

The first, second and third derivative of (2) is given by

$$y'(x) = \sum_{n=0}^{r+s-1} a_n T_n'(x) \tag{6}$$

$$y''(x) = \sum_{n=0}^{r+s-1} a_n T_n''(x) \tag{7}$$

$$y'''(x) = \sum_{n=0}^{r+s-1} a_n T_n'''(x) \tag{8}$$

Where $x \in (a,b)$, the $a_n's$ are the real unknown parameters to be determined, **r** is the number of collocation points, s is the number of interpolation points, **r+s** is the sum of collocation and interpolation points.

By considering two step method with one off step point as shown below, we shall develop the desired block methods.

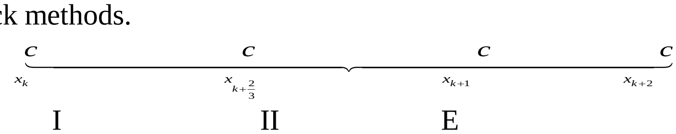

Figure 1: Subdivision of solution interval.

I and E in figure 1 are interpolation and evaluation points respectively while C is the collocation points.

Conventionally, we need to interpolate at at least three points to be able to approximate the solution to (1) and, for this purpose, we proceed by arbitrarily selecting an off step point, x_{k+v}, $v \in (0,1)$ in (x_k, x_{k+2}) in such a manner that the zero-stability of the main method is guaranteed. Then (ii) is interpolated at x_{k+i}, i=0, $^2/_3$ and 1 and its third derivative is collocated at x_{k+i} i=0, $^2/_3$, 1 and 2, so as to obtain a system of 7 equations each of degree six i.e. r+s-1=6 as follows:

$$\sum_{n=0}^{6} a_n T_n(x) = y(x) \tag{9}$$

$$\sum_{n=0}^{6} a_n T_n'''(x) = f(x, y, y', y'') \tag{10}$$

Collocating (10) at $x = x_{k+i}$, $i = 0, \frac{2}{3}, 1$ and 2, and interpolating (9) at $x = x_{k+i}$, $i = 0, \frac{2}{3}$ and 1 lead to the matrix equation:

$$\begin{pmatrix} 0 & 0 & 0 & 192 & -1536 & 6720 & -21504 \\ 0 & 0 & 0 & 192 & 512 & -320/3 & -17408/9 \\ 0 & 0 & 0 & 192 & 1536 & 6720 & 21504 \\ 0 & 0 & 0 & 192 & 4608 & 68160 & 21504 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1/3 & -7/9 & -23/27 & 17/81 & 241/243 & 329/729 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} h^3 f_k \\ h^3 f_{k+2/3} \\ h^3 f_{k+1} \\ h^3 f_{k+2} \\ y_k \\ y_{k+2/3} \\ y_{k+1} \end{pmatrix} \tag{11}$$

Equation (11) is solved by using Maple software to obtain the value of the unknown parameters
a_j ,j=0, 1,…,6 as follows:

$$a_0 = \frac{8513}{13271040} h^3 f_k + \frac{11807}{2949120} h^3 f_{k+\frac{2}{3}} - \frac{4087}{3317760} h^3 f_{k+1} + \frac{1567}{26542080} h^3 f_{k+2} + \frac{5}{16} y_k + \frac{9}{16} y_{k+\frac{2}{3}} + \frac{1}{8} y_{k+1}$$

$$a_1 = -\frac{33}{40960} h^3 f_k - \frac{459}{81920} h^3 f_{k+\frac{2}{3}} + \frac{13}{10240} h^3 f_{k+1} - \frac{17}{245760} h^3 f_{k+2} - \frac{1}{2} y_k + \frac{1}{2} y_{k+1}$$

$$a_2 = -\frac{9439}{26542080} h^3 f_k - \frac{22561}{5898240} h^3 f_{k+\frac{2}{3}} + \frac{4961}{6635520} h^3 f_{k+1} - \frac{2081}{53084160} h^3 f_{k+2} + \frac{3}{16} y_k - \frac{9}{16} y_{k+\frac{2}{3}} + \frac{3}{8} y_{k+1}$$

$$a_3 = \frac{37}{49152} h^3 f_k + \frac{189}{32768} h^3 f_{k+\frac{2}{3}} - \frac{17}{12288} h^3 f_{k+1} + \frac{7}{98304} h^3 f_{k+2}$$

$$a_4 = -\frac{139}{491520} h^3 f_k - \frac{63}{327680} h^3 f_{k+\frac{2}{3}} + \frac{61}{122880} h^3 f_{k+1} - \frac{7}{327680} h^3 f_{k+2}$$

$$a_5 = \frac{13}{245760} h^3 f_k - \frac{27}{163840} h^3 f_{k+\frac{2}{3}} + \frac{7}{61440} h^3 f_{k+1} - \frac{1}{491520} h^3 f_{k+2}$$

$$a_6 = -\frac{1}{327680} h^3 f_k + \frac{9}{655360} h^3 f_{k+\frac{2}{3}} - \frac{1}{81920} h^3 f_{k+1} + \frac{1}{655360} h^3 f_{k+2} \tag{12}$$

Substituting (7) into (2) yields a continuous implicit hybrid two –step method in the form

$$y(x) = \sum_{j=0}^{1} \alpha_j y_{n+j} + \alpha_{\frac{2}{3}}(x) y_{n+\frac{2}{3}} + h^3 \left(\sum_{j=0}^{2} \beta_j(x) f_{n+j} + \beta_{\frac{2}{3}}(x) f_{n+\frac{2}{3}} \right) \tag{13}$$

Where $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients,

$$y_{k+j} \equiv y(x_{k+j}) \equiv y(x_k + jh)$$

is the numerical approximation of the analytical solution at x_{n+j} and

$$f_{k+j} = f(x_{k+j}, y_{k+j}, y'_{k+j}, y''_{k+j}).$$

Equation (13) yields the parameters α_j and β_j as the following continuous functions of t.

$$\left.\begin{array}{l}
\alpha_0(t) = \frac{3}{8}t^2 - \frac{1}{2}t + \frac{1}{8} \\
\alpha_{\frac{2}{3}}(t) = \frac{9}{8} - \frac{9}{8}t^2 \\
\alpha_1(t) = \frac{3}{4}t^2 + \frac{1}{2}t - \frac{1}{4} \\
\beta 0(t) = \frac{119}{165888} - \frac{43}{15360}t + \frac{1241}{829440}t^2 + \frac{1}{512}t^3 - \frac{13}{6144}t^4 + \frac{13}{15360}t^5 - \frac{1}{10240}t^6 \\
\beta_{\frac{2}{3}}(t) = \frac{281}{36864} - \frac{243}{10240}t - \frac{1081}{184320}t^2 + \frac{27}{1024}t^3 - \frac{9}{4096}t^4 - \frac{27}{10240}t^5 + \frac{9}{20480}t^6 \\
\beta_1(t) = -\frac{61}{41472} + \frac{23}{3840}t - \frac{559}{207360}t^2 - \frac{1}{128}t^3 + \frac{7}{1536}t^4 + \frac{7}{3840}t^5 - \frac{1}{2560}t^6 \\
\beta_2(t) = \frac{25}{331776} - \frac{3}{10240}t + \frac{199}{1658880}t^2 + \frac{1}{3072}t^3 - \frac{1}{4096}t^4 - \frac{1}{30720}t^5 + \frac{1}{20480}t^6
\end{array}\right\} \quad (14)$$

where $t = \frac{2x - 2x_k - h}{h}$

Evaluating (13) at x_{k+2}, the main method is obtained as

$$y_{k+2} = 2y_k - 9y_{k+\frac{2}{3}} + 8y_{k+1} + \frac{h^3}{648}\left(14f_k + 63f_{k+\frac{2}{3}} + 200f_{k+1} + 11f_{k+2}\right) \quad (15)$$

The scheme is of order $P = 4$; and the error constant is $C_{p+3} = -\frac{31}{29160}$

The first derivatives of continuous functions are given as

$$\left.\begin{array}{l}
\alpha_0'(t) = \frac{3}{2h}t - 1 \\
\alpha_{\frac{2}{3}}'(t) = -\frac{9}{2h}t \\
\alpha_1'(t) = \frac{3}{h}t + 1 \\
\beta_0'(t) = \frac{1}{h}\left(-\frac{43}{7680} + \frac{1241}{207360}t + \frac{3}{256}t^2 - \frac{13}{768}t^3 + \frac{13}{1536}t^4 - \frac{3}{2560}t^5\right) \\
\beta_{\frac{2}{3}}'(t) = \frac{1}{h}\left(-\frac{243}{5120} - \frac{1081}{46080}t + \frac{81}{512}t^2 - \frac{9}{512}t^3 - \frac{27}{1024}t^4 + \frac{27}{5120}t^5\right) \\
\beta_1'(t) = \frac{1}{h}\left(\frac{23}{1920} - \frac{559}{51840}t - \frac{3}{64}t^2 + \frac{7}{192}t^3 + \frac{7}{384}t^4 - \frac{3}{640}t^5\right) \\
\beta_2'(t) = \frac{1}{h}\left(-\frac{3}{5120} + \frac{199}{414720}t + \frac{1}{512}t^2 - \frac{1}{512}t^3 - \frac{1}{3072}t^4 + \frac{3}{5120}t^5\right)
\end{array}\right\} \quad (16)$$

The second derivatives of continuous functions are given as

$$\alpha_0''(t) = \frac{3}{h^2}$$
$$\alpha_{\frac{2}{3}}''(t) = -\frac{9}{h^2} \tag{17}$$
$$\alpha_1''(t) = \frac{6}{h^2}$$
$$\beta_0''(t) = \frac{1}{h^2}\left(\frac{1241}{103680} + \frac{3}{64}t - \frac{13}{128}t^2 + \frac{13}{192}t^3 - \frac{3}{256}t^4\right)$$
$$\beta_{\frac{2}{3}}''(t) = \frac{1}{h^2}\left(-\frac{1081}{23040} + \frac{81}{128}t - \frac{27}{256}t^2 - \frac{27}{128}t^3 + \frac{27}{512}t^4\right)$$
$$\beta_1''(t) = \frac{1}{h^2}\left(-\frac{559}{25920} - \frac{3}{16}t + \frac{7}{32}t^2 + \frac{7}{48}t^3 - \frac{3}{64}t^4\right)$$
$$\beta_2''(t) = \frac{1}{h^2}\left(\frac{199}{207360} + \frac{1}{128}t - \frac{3}{256}t^2 - \frac{1}{384}t^3 + \frac{3}{512}t^4\right)$$

The additional methods to be coupled with the main method are obtained by evaluating (16) and (17) at x_k, $x_{k+\frac{2}{3}}$, x_{k+1} and x_{k+2} respectively. This yields the following discrete derivative schemes:

$$hy_k' + \frac{5}{2}y_k - \frac{9}{2}y_{k+\frac{2}{3}} + 2y_{k+1} = h^3\left(\frac{173}{6480}f_k + \frac{173}{1440}f_{k+\frac{2}{3}} - \frac{61}{1620}f_{k+1} + \frac{5}{2592}f_{k+2}\right) \tag{18}$$
$$hy_{k+\frac{2}{3}}' + \frac{1}{2}y_k + \frac{3}{2}y_{k+\frac{2}{3}} - 2y_{k+1} = h^3\left(-\frac{11}{3888}f_k - \frac{167}{4320}f_{k+\frac{2}{3}} + \frac{23}{4860}f_{k+1} - \frac{11}{38880}f_{k+2}\right) \tag{19}$$
$$hy_{k+1}' - \frac{1}{2}y_k + \frac{9}{2}y_{k+\frac{2}{3}} - 4y_{k+1} = h^3\left(\frac{1}{405}f_k + \frac{7}{144}f_{k+\frac{2}{3}} + \frac{7}{1620}f_{k+1} + \frac{1}{6480}f_{k+2}\right) \tag{20}$$
$$hy_{k+2}' - \frac{7}{2}y_k + \frac{27}{2}y_{k+\frac{2}{3}} - 10y_{k+1} = h^3\left(\frac{133}{2160}f_k - \frac{11}{480}f_{k+\frac{2}{3}} + \frac{95}{108}f_{k+1} + \frac{353}{4320}f_{k+2}\right) \tag{21}$$
$$h^2y_k'' - 3y_k + 9y_{k+\frac{2}{3}} - 6y_{k+1} = h^3\left(\frac{1399}{6480}f_k - \frac{751}{1440}f_{k+\frac{2}{3}} + \frac{311}{1620}f_{k+1} - \frac{131}{12960}f_{k+2}\right) \tag{22}$$
$$h^2y_{k+\frac{2}{3}}'' - 3y_k + 9y_{k+\frac{2}{3}} - 6y_{k+1} = h^3\left(\frac{121}{6480}f_k + \frac{209}{1440}f_{k+\frac{2}{3}} - \frac{89}{1620}f_{k+1} + \frac{29}{12960}f_{k+2}\right) \tag{23}$$
$$h^2y_{k+1}'' - 3y_k + 9y_{k+\frac{2}{3}} - 6y_{k+1} = h^3\left(\frac{43}{3240}f_k + \frac{29}{90}f_{k+\frac{2}{3}} + \frac{44}{405}f_{k+1} + \frac{1}{3240}f_{k+2}\right) \tag{24}$$
$$h^2y_{k+2}'' - 3y_k + 9y_{k+\frac{2}{3}} - 6y_{k+1} = h^3\left(\frac{761}{6480}f_k - \frac{751}{1440}f_{k+\frac{2}{3}} + \frac{2471}{1620}f_{k+1} + \frac{4189}{12960}f_{k+2}\right) \tag{25}$$

Equations (15),(18) ,(19) ,(20) , (21) ,(22) ,(23) ,(24) and (25) are solved using Shampine and Watts(1969) block formula defined as

$$Ay_m = hBF(y_m) + ey_n + hdf_n \tag{26}$$

where $A = (a_{ij})$, $B = (b_{ij})$, column vectors $e = (e_1 \ldots e_r)^T$, $d = (d_1 \ldots d_r)^T$

$y_m = (y_{n+1} \ldots y_{n+r})^T$ and $F(y_m) = (f_{n+1} \ldots f_{n+r})^T$.

According to (23)…(25) A, B, d and e are obtained from Shampine equation (26) as follow;

$$A=\begin{pmatrix} 9 & -8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{9}{2} & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{2} & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{9}{2} & -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{27}{2} & -10 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 9 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & -6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 9 & -6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 9 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B=\begin{pmatrix} \frac{7}{72} & \frac{25}{81} & \frac{11}{648} \\ \frac{173}{1440} & -\frac{61}{1620} & \frac{5}{2592} \\ -\frac{167}{4320} & \frac{23}{4860} & -\frac{11}{38880} \\ \frac{7}{144} & \frac{7}{1620} & \frac{1}{6480} \\ -\frac{11}{480} & \frac{95}{108} & \frac{353}{4320} \\ -\frac{751}{1440} & \frac{311}{1620} & -\frac{131}{12960} \\ \frac{209}{1440} & -\frac{89}{1620} & \frac{29}{12960} \\ \frac{29}{90} & \frac{44}{405} & \frac{1}{3240} \\ -\frac{751}{1440} & \frac{2471}{1620} & \frac{4189}{12960} \end{pmatrix} \qquad E=\begin{pmatrix} 2 & 0 & 0 \\ -\frac{5}{2} & -1 & 0 \\ -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{7}{2} & 0 & 0 \\ 3 & 0 & -1 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

$$D=\begin{pmatrix} \frac{7}{324} & \frac{173}{6480} & -\frac{11}{3888} & \frac{1}{405} & \frac{133}{2160} & -\frac{1399}{6480} & \frac{121}{6480} & \frac{43}{3240} & \frac{761}{6480} \end{pmatrix}^T$$

Substituting A, B, D and E into (26) we obtain the explicit schemes;

$$\begin{aligned}
y_{k+\frac{2}{3}} &= y_k + \frac{2}{3}hy'_k + \frac{2}{9}h^2 y''_k + \frac{22}{729}h^3 f_k + \frac{29}{810}h^3 f_{k+\frac{2}{3}} - \frac{64}{3645}h^3 f_{k+1} + \frac{7}{7290}h^3 f_{k+2} \\
y_{k+1} &= y_k + hy'_k + \frac{1}{2}h^2 y''_k + \frac{13}{160}h^3 f_k + \frac{9}{64}h^3 f_{k+\frac{2}{3}} - \frac{7}{120}h^3 f_{k+1} + \frac{1}{320}h^3 f_{k+2} \\
y_{k+2} &= y_k + 2hy'_k + 2h^2 y''_k + \frac{2}{5}h^3 f_k + \frac{9}{10}h^3 f_{k+\frac{2}{3}} + \frac{1}{30}h^3 f_{k+2} \\
y'_{k+\frac{2}{3}} &= hy'_k + \frac{2}{3}h^2 y''_k + \frac{139}{1215}h^3 f_k + \frac{17}{90}h^3 f_{k+\frac{2}{3}} - \frac{104}{1215}h^3 f_{k+1} + \frac{11}{2430}h^3 f_{k+2} \\
y'_{k+1} &= hy'_k + h^2 y''_k + \frac{23}{120}h^3 f_k + \frac{9}{20}h^3 f_{k+\frac{2}{3}} - \frac{3}{20}h^3 f_{k+1} + \frac{1}{120}h^3 f_{k+2} \\
y'_{k+2} &= hy'_k + 2h^2 y''_k + \frac{7}{15}h^3 f_k + \frac{9}{10}h^3 f_{k+\frac{2}{3}} + \frac{8}{15}h^3 f_{k+1} + \frac{1}{10}h^3 f_{k+2} \\
y''_{k+\frac{2}{3}} &= h^2 y''_k + \frac{19}{81}h^3 f_k + \frac{2}{3}h^3 f_{k+\frac{2}{3}} - \frac{20}{81}h^3 f_{k+1} + \frac{1}{81}h^3 f_{k+2} \\
y''_{k+1} &= h^2 y''_k + \frac{11}{48}h^3 f_k + \frac{27}{32}h^3 f_{k+\frac{2}{3}} - \frac{1}{12}h^3 f_{k+1} + \frac{1}{96}h^3 f_{k+2} \qquad (27) \\
y''_{k+2} &= h^2 y''_k + \frac{1}{3}h^3 f_k + \frac{4}{3}h^3 f_{k+1} + \frac{1}{3}h^3 f_{k+2}
\end{aligned}$$

The block formulae are all of order $P=4$; with the error constants $C_{P+3}=-\frac{44}{229635},-\frac{37}{60480},-\frac{4}{945},-\frac{7}{4320},-\frac{1}{135},-\frac{29}{32805},-\frac{17}{7290},-\frac{1}{480},-\frac{1}{90}$ respectively.

ANALYSIS OF THE METHOD

The basic properties of this method such as order, error constant, zero stability and consistency are analysed hereunder.

Equation (15) derived is a discrete scheme belonging to the class of LMMs of the form

$$\sum_{j=0}^{k}\alpha_j y_{n+j}=h^3\sum_{j=0}^{k}\beta_j f_{n+j} \qquad (28)$$

Following Fatunla (1988) and Lambert (1973), we define the local truncation error associated with equation (28) by the difference operator

$$L[y(x):h]=\sum_{j=0}^{k}\left[\alpha_j y(x_n+jh)-h^3\beta_j f(x_n+jh)\right] \qquad (29)$$

Where $y(x)$ is an arbitrary function, continuously differentiable on [a, b].

Expanding (29) in Taylor series about the point x, we obtain the expression

$$L[y(x);h]=C_0 y(x)+C_1 hy'(x)+C_2h^2y''(x)+\ldots+C_{p+3}h^{p+3}y^{p+3}(x)$$

Where the C_0 , C_1 , $C_2 \ldots C_p \ldots C_{p+2}$ are obtained as

$$C_0=\sum_{j=0}^{k}\alpha_j$$

$$C_1=\sum_{j=1}^{k}j\alpha_j$$

$$C_2=\frac{1}{2!}\sum_{j=1}^{k}j^2\alpha_j$$

$$C_q=\frac{1}{q!}\left[\sum_{j=1}^{k}j^q\alpha_j-q(q-1)(q-2)\sum_{j=1}^{k}\beta_j j^{q-3}\right]$$

In the sense of Lambert (1973), equation (28) is of order P if

$C_0=C_1=C_2=\ldots C_p=C_{p+2}=0$ and $C_{p+3}\neq 0$

The $C_{p+3}\neq 0$ is called the error constant and $C_{p+3}h^{p+3}y^{p+3}(x_n)$ is the principal local truncation error at the point x_n.

Using the concept above, (15) has order $P=4$ and error constant given by $C_{p+3}=-\frac{31}{29160}$

ZERO STABILITY OF THE METHOD

The linear multistep method (28) is said to be zero-stable if no root of the first characteristic polynomial $\rho(R)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation.

To analyse the zero-stability of the method, we present (27) in vector notation form of column vectors $e=(e_1\ldots e_r)^T$, $d=(d_1\ldots d_r)^T$, $y_m=(y_{n+1}\ldots y_{n+r})^T$, $F(y_m)=(f_{n+1}\ldots f_{n+r})^T$ and matrices $A=(a_{ij})$, $B=(b_{ij})$.

Thus, equation (27) forms the block formula $A^0y_m=hBF(y_m)+A^1y_n+hdf_n$

Where h is a fixed mesh size within a block.

In line with this,

$$A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{k+\frac{2}{3}} \\ y_{k+1} \\ y_{k+2} \end{pmatrix} \qquad A^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{k-2} \\ y_{k-1} \\ y_k \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{29}{810} & -\frac{64}{3645} & \frac{7}{7290} \\ \frac{9}{64} & -\frac{7}{120} & \frac{1}{320} \\ \frac{9}{10} & 0 & \frac{1}{30} \end{pmatrix}\begin{pmatrix} f_{k+\frac{2}{3}} \\ f_{k+1} \\ f_{k+2} \end{pmatrix} \qquad d = \begin{pmatrix} \frac{22}{729} \\ \frac{13}{160} \\ \frac{2}{5} \end{pmatrix}\begin{pmatrix} f_{k-2} \\ f_{k-1} \\ f_k \end{pmatrix}$$

The first characteristic polynomial of the block hybrid method is given by

$$\rho(R) = \det(RA^0 - A^1) \qquad (30)$$

Where

$$A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$A^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

substituting A^0 and A^1 in equation (30) and solving for R, the values of R are obtained as 0 and 1.

According to Fatunla (1988, 1991), the block method equation (27) are zero-stable, since from (30), $\rho(R) = 0$, satisfy $\left|R_j\right| \leq 1$, $j = 1$ and for those roots with $\left|R_j\right| = 1$, the multiplicity does not exceed three.

CONSISTENCY OF THE METHOD

The linear multistep method (28) is said to be consistent if it has order $P \geq 1$ and the first and second characteristic polynomials which are defined as $\rho(R) = \sum_{j=0}^{k} \alpha_j R^j$ and $\sigma(R) = \sum_{j=0}^{k} \beta_j R^j$ respectivelywhere R ,the principal root satisfies the following conditions

(i) $\sum_{j=0}^{k} \alpha_j = 0$ (ii) $\rho(1) = \rho'(1) = 0$ (iii) $\rho'''(1) = 3!\sigma(1)$

The scheme (15) derived is of order $P = 5 > 1$ and it has been investigated to satisfy conditions (i)$\ldots$ (iii). Hence the scheme is consistent.

CONVERGENCY OF THE METHOD

According to the theorem of Dahlquist; the necessary and sufficient condition for a LMM to be convergent is to be consistent and zero stable. Since the method satisfies the two conditions hence it is convergent.

NUMERICAL EXAMPLES

We consider here four test problems to illustrate the method.

Problem 1: (A constant coefficient homogeneous problem)

$$y'''+y'=0 \qquad y(0)=0,\ \ y'(0)=1,\ \ y''(0)=2$$

h= 0.1

Exact solution: $y(x)=2(1-Cosx)+Sinx$

Source: Anake*et al*. (2013)

Problem 2: (A constant coefficient non homogeneous problem)

$$y'''+y''+3y'-5y=2+6x-5x^2$$

$$y(0)=-1,\ y'(0)=1,\ y''(0)=-3 \quad 0\le x\le 1$$

Exact solution: $y(x)=x^2-e^x+e^{-x}\sin(2x)$.

Source: Awoyemi et *al.* (2014)

Problem 3: (A variable coefficient singular problem)

$$y'''+\frac{\cos x}{\sin x}y''=\sin x\cos x \qquad y(0)=1, y'(0)=-2,\ y''(0)=0, h=0.1$$

Exact solution: $y(x)=1-2x+\frac{x^2}{12}-\frac{\sin^2 x}{12}$

Problem 4: (A non-linear problem)

$$y^2y'''=1 \qquad y(0)=1,\ y'(0)=1,\ y''(0)=1, h=0.1$$

Source: Awoyemi et *al.* (2014)

The above problem was derived by Tanner (1979) to investigate the motion of the contact line for a thin oil drop spreading on a horizontal surface.

Table of Results

TABLE 1: Numerical Results for Problem 1.

X	EXACT SOLUTION	NEW RESULT	ERROR IN NEW RESULT	ANAKE BLOCK ALGORITHM [13]	ERROR IN ANAKE BLOCK ALGORITHM [13]
0.1	0.109825086	0.109825086	9.070000000D-11	0.109825087	1.608800000D-09
0.2	0.238536175	0.238536175	4.125000000D-10	0.238536188	1.038700000D-08
0.3	0.384847228	0.384847227	1.243859872D-09	0.384847257	2.957200000D-08
0.4	0.547296354	0.547296351	3.402878300D-09	0.547296585	2.314700000D-07
0.5	0.724260414	0.724260408	6.623457000D-09	0.724259960	4.542000000D-07
0.6	0.913971243	0.913971232	1.147567740D-08	0.913969778	1.474600000D-07
0.7	1.114533313	1.114533294	1.866871200D-08	1.114530439	2.873400000D-06
0.8	1.323942672	1.323942644	2.820519300D-08	1.323937959	4.682600000D-06
0.9	1.540106973	1.540106933	4.008615600D-08	1.540100051	6.921700000D-06
1.0	1.760866373	1.760866318	5.507161900D-08	1.760856775	9.597400000D-06

Table of Results

TABLE 2: Numerical Result for Problem 2.

X	EXACT SOLUTION	NEW RESULT K=2 , P=4	ERROR IN AWOYEMI(2014) K=4 , P=7	ERROR IN NEW RESULT
0.1	-0.915407473	-0.915407459	8.547820000E-11	1.380000000E-08
0.2	-0.862573985	-0.862573898	2.232510000E-09	8.690000000E-07
0.3	-0.841561375	-0.841561125	5.824412000E-08	2.492000000E-07
0.4	-0.850966529	-0.850966011	1.226405000E-06	5.173000000E-07
0.5	-0.888343319	-0.888342419	2.811820000E-06	8.993000000E-07
0.6	-0.950604904	-0.950603515	6.295841000E-06	1.388900000E-06
0.7	-1.034392854	-1.034390877	1.695782000E-05	2.645000000E-06
0.8	-1.136403557	-1.136400912	4.765221000E-05	2.645000000E-06
0.9	-1.253666211	-1.253662838	1.316541000E-04	3.373000000E-06
1.0	-1.383769999	-1.383765856	3.417856000E-04	4.143000000E-06

Table of Results

TABLE 3: Error of Method for Problem 3.

X	EXACT RESULT	NEW RESULT	ERROR IN NEW RESULT
0.1	0.800002774	0.800002774	2.328160000E-11
0.2	0.600044208	0.600044208	3.000000000E-10
0.3	0.400222317	0.400222318	1.100000000E-09
0.4	0.200696112	0.200696115	3.100000000E-09
0.5	0.001679263	0.001679268	4.900494170E-09
0.6	-0.196568426	-0.196568417	8.600000000E-09
0.7	-0.393751369	-0.393751352	1.660000000E-08
0.8	-0.589549980	-0.589549953	2.700000000E-08
0.9	-0.783633420	-0.783633379	4.090000000E-08
1.0	-0.975672784	-0.975672724	5.970000000E-08

Table of Results

TABLE 4: Numerical Result for Problem 4.

X	EXACT SOLUTION	NEW RESULT	ERROR IN NEW RESULT	ERROR IN TANNER PROBLEM [12]
0.1	*	1.105158657	*	*
0.2	1.221211030	1.221210010	1.0200000E-06	2.4050000E-05
0.3	*	1.348898706	*	*
0.4	1.488834893	1.488834813	8.0000000E-08	7.7167000E-05
0.5	*	1.641518699	*	*
0.6	1.807361404	1.807361492	8.8000000E-08	7.9494500E-06
0.7	*	1.986701820	*	*
0.8	2.179819234	2.179819431	1.9700000E-07	4.3494900E-03
0.9	*	2.386946234	*	*
1.0	2.608275822	2.608275216	6.0600000E-07	1.8319962E-02

The result test problem 8 at $x \in [0.2, 0.4, 0.6, 0.8, 1.0]$

* Not available for comparison.

CONCLUSION

The derivation of continuous numerical integration schemes for the class of Initial Value Problems in third order ordinary differential equation has been presented.

These schemes are in the block form and consequently they do not require other method (especially one-step methods) in order to implement them.

The method was applied to four problems, each with its own peculiarity and the results obtained demonstrate their effectiveness and accuracy viz-a-viz some other existing schemes (Anake (2013), Awoyemi (2014), Tanner (1979)) as this new method performs favourably well.

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