A PARTIAL BACKLOGGING INVENTORY MODEL FOR DETERIORATING ITEM WITH STOCK-DEPENDENT DEMAND AND TIME VARYING HOLDING COST

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Abstract
This model investigates instantaneous deterioration rate with linear time dependent holding cost and partial backlogging rate. Most literature on deterministic inventory models considered holding cost to be constant while others considered variable holding cost with complete backlogging where deterioration is non-instantaneous. In real life situation, deterioration of some items starts immediately the stock arrives and also backlogging cannot always be complete due to dynamic nature of human being. The inventory policy is to determine the optimal replenishment policy in order to maximize the profit function. The necessary and sufficient conditions for optimality were determined. Then Numerical examples with the help of Newton Raphson iteration method and sensitivity analysis were carried out to show the workability of the model. It has been found out from the results of the model that the changes in stock dependent demand rate, holding cost, deterioration rate, replenishment size, shortage cost and partial backlogging rate do have a great impact on the optimal replenishment policy.

Keywords: Deterioration, partial backlogging rate, shortage cost

1 Introduction
In classical inventory model the demand rate is assumed to be a constant. In the real sense demand for physical goods may be time dependent, price dependent and stock dependent. In most cases the demand rate is directly related to the amount of inventory displayed, most especially, the demand for certain consumer goods. As observed by Levin et al. (1972), “it is a common belief that large piles of goods displayed in a supermarket will lead customers to buy more.” Gupta and Vrat (1986) developed a model for stock dependent demand where the demand for the products is a function of initial stock level and the model was analyzed through cost minimization. Datta and Pal (1990) developed an inventory model with inventory level dependent demand rate, where they discussed an infinite time horizon deterministic inventory model without shortages and the demand rate at any instant depends on the on-hand inventory. Urban (1992) presented an inventory model with an inventory level dependent demand and relaxed terminal conditions. Datta et al. (1988) proposed a model on stock dependent demand situation; the model was analyzed through profit maximization. Datta and Paul (2001) developed an inventory system with stock-dependent, price sensitive demand rate. Urban (2005) analyzed an inventory model with inventory level-dependent demand: a comprehensive review and unifying theory. Pal and Manisha (2014) analysed an inventory model with stock-dependent demand, permissible delay in payment and price discount on backorder. The effect of deterioration in our day to day activities cannot be ignored in investigating an inventory model. Decay or deterioration plays an important role in inventory system. Many physical goods undergo decay or deterioration with change in time. For example, fruits, vegetables and food items undergo depletion by direct spoilage while stored. Electronic goods, radioactive substances, photographic film, grain, medicine, blood banks and so on deteriorate through a gradual loss of potential or utility over time. Whitin (1957) studied deterioration of the fashion goods at the end of a perishable shortage period. Ghar and Schrada (1963) considered a model for exponentially decaying inventory. Agarwal (1978) developed a note on an order level inventory model for a system with constant deterioration. Dave and Patel (1981) considered inventory model for...

Most of the researches on inventory models consider holding cost to be constant. But in real life situation, holding cost cannot always be constant because of the time value of money; prices of items increase over time. Weiss (1982) developed an economic order quantity model with nonlinear holding cost. Goh (1994) considered holding cost with two possibilities of variation; (a) a non-linear function of the length of time the item is held in the inventory, (b) a non-linear function of the amount of on-hand inventory. Giri and Chaudhuri (1998) developed a deterministic models of perishable inventory model with stock-dependent demand rate and non-linear holding cost. Chang (2004) studied an inventory model with stock-dependent demand and non-linear holding cost. Alfares (2007) discussed inventory model with stock dependent consumption rate and variable holding cost. Sahoo et al. (2010) investigated time varying holding cost model with constant deterioration. An optimal replenishment policy for non-instantaneous deteriorating items with stock dependent demand and variable holding cost was studied by Tyagi et al. (2014). Dutta and Kumar (2015) developed a partial backlogging inventory model for deteriorating items with time-varying demand and holding cost.

In many inventory models with shortages, the unsatisfied demand is either completely lost or completely backlogged. But in real life situation, it is quite possible that while some customers leave, others are willing to wait till fulfillment of their demand. The inability of some customers to wait results to what is called loss of sale. The accumulated demand of customers who are willing to wait is partially backlogged because the demand of those customers who were not willing to wait has been lost already. Chang and Dye (2001) considered an inventory model for deteriorating items with partial backlogging and permissible delay in payment. Wu et al. (2006) developed an optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand and partial backlogging. Chang and Feng (2010) analyzed a partial backlogging inventory model for non-instantaneous deteriorating items with stock dependent consumption rate under inflation. Yang et al. (2010) considered an inventory model under inflation for deteriorating items with stock dependent consumption rate and partial backlogging. Rajeswari and Vanjikkadi (2011) developed a deteriorating inventory model with power demand and partial backlogging. Sarkar and Sarkar (2013) presented an inventory model with partial backlogging, time varying deterioration and stock dependent demand. Sicilia et al. (2014) discussed an inventory model for deteriorating items with shortages and time varying demand where they considered the shortages to be partially backlogged. An inventory model for non-instantaneous deteriorating items with partial backlogging, permissible delay in payments, inflation and selling price-dependent demand.
was considered by Ghoreishi et al. (2015). Khurana et al. (2015) presented a supply chain production inventory model for deteriorating product with stock dependent demand under inflationary environment and partial backlogging. Jaggi et al. (2016) analyzed the effect of inflation and time value of money on an inventory system with deteriorating items and partial backlogged shortages. Preservation of deteriorating seasonal products with stock dependent consumption rate and shortages was presented by Sakar et al. (2017) in which they considered partially backlogged shortages. An optimal inventory replenishment policy for a perishable item with time quadratic demand and partial backlogging with shortages in all cycles was also proposed by Chowdhury et al. (2017).

Choudhury et al. (2013) considered an inventory model for deteriorating item with stock dependent demand, time varying holding cost and shortages with complete backlogging. Most of the papers on deterministic inventory model which were reviewed above, considered holding cost to be constant while others considered variable holding cost with complete backlogging. In this work, we proposed an inventory model for deteriorating item with stock-dependent demand and time varying holding cost where shortages are allowed with partial backlogging rate based on the work of Choudhury et al. (2013). The deterioration starts at the instant of the arrival of the item in the inventory (i.e. instantaneous deterioration) in which the inventory level depletes to zero as a result of both deterioration and market demand. When shortages begin the unsatisfied demand is partially backlogged with a constant demand rate. Some theoretical and analytical results were considered in order to maximize the total average profit. Numerical example was provided to demonstrate the application of the model and also a sensitivity analysis was carried out to study the effects of various changes in some model parameter on the decision variable.

2 Notation and Assumptions

2.1 Notation

\( P \) The selling price of the item per unit
\( C \) The purchasing cost per unit of the item
\( C_2 \) The replenishment cost per order
\( t_1 \) The time at which the inventory level becomes zero
\( C_1 \) The shortage in time per unit
\( T \) The cycle length
\( l \) The lost in time per unit
\( Q \) The replenishment size
\( K \) The maximum inventory level during the time period \( T \)
\( I(t) \) The inventory level at time

2.2 Assumptions

1. \( 0 < \theta < 1 \), is the constant deterioration rate. Deterioration starts at the instant of the arrival of the item in the inventory and there is no repair or replenishment of deteriorated item.
2. \( \alpha + \beta(t), \ 0 < \beta < 1 \), is the linear stock dependent demand, where \( \beta \) is the stock dependent parameter and \( \alpha \) is the positive demand rate
3. \( C_0(t) = q + bt, \ a > 0 \) and \( b > 0 \), is the linear time dependent holding cost.
4. The replenishment takes place in an infinite rate, i.e., replenishment size is finite.
5. Only one stocking point is involved in the system.
6. Shortages are allowed and partially backlogged during the time period \( t_4 \leq t \leq T \).
backlogging parameter $\delta$ is a positive constant, $0 < \delta < 1$, $\delta = 0$ is a case of no shortages and $\delta = 1$ is a case of complete backlogging.

### Mathematical Formulation of the Model

In this model we determine the optimal order quantity in order to maximize profit. At time $t = 0$, the maximum inventory level $K$, depletes gradually due to both demand and deterioration until it falls to zero at time $t = t_1$. The shortages are allowed during the time period $[t_1, T]$ which are partially backlogged and the demand rate is assumed to be constant.

The differential equations governing the behavior of the inventory system are given as

$$\frac{dI(t)}{dt} + \theta I(t) = -\alpha - \beta I(t), \quad 0 \leq t \leq t_1$$  \hspace{1cm} (1)

with boundary conditions $I(t) = K$ at $t = 0$ and $I(t) = 0$ at $t = t_1$

and

$$\frac{dI(t)}{dt} = -\alpha \delta \quad t_1 \leq t \leq T$$  \hspace{1cm} (2)

with condition $I(t) = 0$ at $t = t_1$

![Graphical representation of the inventory system](image_url)

The solution of the differential eq. (1) is given as follows:

Using integrating factor method with $e^{(\theta+\beta)t}$ as our integrating factor we have,

$$I(t) = -\frac{\alpha}{\theta+\beta} + Ke^{-(\theta+\beta)t} + \frac{\alpha}{\theta+\beta} e^{-(\theta+\beta)t}$$

$$= \left( K + \frac{\alpha}{\theta+\beta} \right) e^{-(\theta+\beta)t} - \frac{\alpha}{\theta+\beta}$$  \hspace{1cm} (3)

Using the boundary condition $I(t) = 0$ at $t = t_1$ the solution of differential eq. (2) is

$$I(t) = -\alpha \delta (t - t_1)$$  \hspace{1cm} (4)

in order to obtain the maximum inventory level we use the condition $I(t) = 0$ at $t_1$ in eq. (3) which yields

$$0 = \left( K + \frac{\alpha}{\theta+\beta} \right) e^{-(\theta+\beta)t_1} - \frac{\alpha}{\theta+\beta}$$

$$K = \frac{\alpha}{\theta+\beta} \left( e^{-(\theta+\beta)t_1} - 1 \right)$$  \hspace{1cm} (5)

Substituting eq. (5) in eq. (3), we have

$$I(t) = \left( \frac{\alpha}{\theta+\beta} e^{-(\theta+\beta)t_1} - \frac{\alpha}{\theta+\beta} + \frac{\alpha}{\theta+\beta} \right) e^{-(\theta+\beta)t} - \frac{\alpha}{\theta+\beta}$$

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\[= \frac{\alpha}{\Theta + \beta} e^{-(\Theta + \beta)(t_1 - t)} - \frac{\alpha}{\Theta + \beta} \]
\[= \frac{\alpha}{\Theta + \beta} (e^{-(\Theta + \beta)(t_1 - t)} - 1), \quad 0 \leq t \leq t_1 \]  
(6)

The total holding cost for the inventory cycle is obtained as follows
\[H_c = \int_0^{t_1} (q + bt) l(t) dt, \quad q > 0, b > 0 \]
\[= aq \frac{(\Theta + \beta)}{(\Theta + \beta)^2} (e^{(\Theta + \beta)t_1} - 1) - \frac{ab}{(\Theta + \beta)^2} \left( e^{(\Theta + \beta)t_1} - 1 \right) - \frac{ab}{2(\Theta + \beta)} t_1^2 \]
\[= \xi_1 (e^{(\Theta + \beta)t_1} - 1) - \xi_2 t_1 - \xi_3 (e^{(\Theta + \beta)t_1} - 1) - \xi_4 t_1^2 \]  
(7)

where
\[\xi_1 = \frac{aq}{(\Theta + \beta)^2}, \quad \xi_2 = \frac{a(\Theta + \beta + b)}{(\Theta + \beta)^2}, \quad \xi_3 = \frac{ab}{(\Theta + \beta)^3}, \quad \xi_4 = \frac{ab}{2(\Theta + \beta)} \]

The total amount of deterioration during this cycle
\[D_c = \Theta \int_{t_1}^{T} l(t) dt \]
\[= a\frac{\theta}{(\Theta + \beta)^2} (e^{(\Theta + \beta)t_1} - 1) - \frac{a\theta t_1}{\Theta + \beta} \]  
(8)

The shortage cost during this cycle is given as
\[S_c = \int_0^{t_1} a\delta (t - t_1) dt \]
\[= C_1 a\delta \frac{(T-t_1)^2}{2} \]  
(9)

The total amount backordered is given by
\[B = Q - K = a\delta(T - t_1) \]  
(10)

The lost sale during this cycle is given by
\[S_c = l \int_{t_1}^{T} a(1 - \delta) dt \]
\[= la(1 - \delta)(T - t_1) \]  
(11)

Now, we obtain the profit function for the replenishment cycle per unit time as follows
\[g(t_1, T) = \frac{1}{T} \left[ Sales Revenue - Ordering Cost - Holding Cost - Shortage Cost - Backordering cost - Loss sales \right] \]
\[= \frac{1}{T} \left[ (K - D_c) - \frac{1}{T} (KC) - \frac{1}{T} (H_c) - \frac{1}{T} (S_c) - \frac{1}{T} (C_2) - \frac{1}{T} (P - C)B - \frac{1}{T} (l\alpha(1 - \delta)(T - t_1)) \right] \]
\[= \frac{1}{T} \left[ \frac{\alpha}{(\Theta + \beta)^2} (e^{(\Theta + \beta)t_1} - 1) - \frac{\alpha\theta}{(\Theta + \beta)^2} (e^{(\Theta + \beta)t_1} - 1) + \frac{\alpha\theta t_1}{\Theta + \beta} \right] - \frac{1}{T} \left[ \frac{\alpha}{(\Theta + \beta)^2} (e^{(\Theta + \beta)t_1} - 1)C \right.
\[+ \frac{1}{T} \left[ \xi_1 (e^{(\Theta + \beta)t_1} - 1) - \xi_2 t_1 - \xi_3 (e^{(\Theta + \beta)t_1} - 1) - \xi_4 t_1^2 \right] - \frac{1}{T} \left( C_1 a\delta \frac{(T-t_1)^2}{2} \right) - \frac{1}{T} (C_2) +
\[+ \frac{1}{T} a\delta (P - C)(T - t_1) - \frac{1}{T} (l\alpha(1 - \delta)(T - t_1)) \right] = \frac{1}{T} (P - C) \left[ \frac{a}{(\Theta + \beta)^2} (e^{(\Theta + \beta)t_1} - 1) \right] -
\[
\frac{1}{7} \left[ \frac{\alpha \theta \beta + (\theta + \beta)^2 (\xi_1 - \xi_3)}{(\theta + \beta)^2} (e^{(\theta + \beta)t_1} - 1) \right] + \frac{\alpha \theta \beta + (\xi_2 + (\theta + \beta))}{T(\theta + \beta)} + \frac{\xi_4 t_1^2}{T} - \frac{1}{T} \left[ \frac{1}{T} C_1 \alpha \delta (T - t_1)^2 \right] - \frac{1}{T} (C_2) + \frac{1}{T} \alpha \delta (P - C)(T - t_1) - \frac{1}{T} (\lambda \alpha (1 - \delta)(T - t_1))
\]

Thus, the objective function of the inventory problem is:

Maximize \( g(t_1, T) \)

subject to: \( t_1 > 0 \) and \( T > t_1 \)

### 4 Optimal Decision

The necessary conditions for the existence of optimum values are given by

\[
\frac{\partial g}{\partial t_1} = 0 \quad \text{and} \quad \frac{\partial g}{\partial T} = 0 \tag{13}
\]

The sufficiency conditions for the existence of the optimum values are given by

\[
\frac{\partial^2 g}{\partial t_1^2} < 0, \quad \frac{\partial^2 g}{\partial T^2} < 0 \quad \text{and} \quad \frac{\partial^2 g}{\partial t_1 \partial T} > 0 \tag{14}
\]

Now we solve the equation, \( \frac{\partial g}{\partial t_1} = 0 \) as follows:

Setting the derivative of \( g(t_1, T) \) with respect to \( t_1 \) equal to zero, we have

\[
\frac{\partial g}{\partial t_1} = \frac{1}{T} \left[ \alpha (P - C) e^{(\theta + \beta)t_1} \right] - \frac{1}{T} \left[ \frac{\alpha \theta \beta + (\theta + \beta)^2 (\xi_1 - \xi_3)}{(\theta + \beta)^2} e^{(\theta + \beta)t_1} \right] + \frac{\alpha \theta \beta + (\xi_2 + (\theta + \beta))}{T(\theta + \beta)} + \frac{2 \xi_4 t_1}{T} + \frac{T}{C_1 \alpha \delta (T - t_1)} - \frac{1}{T} \alpha \delta (P - C) - \frac{1}{T} (\lambda \alpha (1 - \delta)) = 0
\]

On simplifying, we get

\[
\sigma_1 e^{(\theta + \beta)t_1} + \sigma_2 T - \sigma_3 t_1 + \sigma_4 = 0 \tag{15}
\]

where,
\[
\sigma_1 = \alpha (\theta + \beta)(P - C) - \alpha \theta \beta - (\theta + \beta)^2 (\xi_1 - \xi_3)
\]
\[
\sigma_2 = \alpha \delta C_1 (\theta + \beta)
\]
\[
\sigma_3 = 2 \xi_4 (\theta + \beta) - \alpha \delta C_1 (\theta + \beta)
\]
\[
\sigma_4 = \alpha \theta \beta + (\theta + \beta)^2 (\xi_2 + (\theta + \beta)) (P - C) + \alpha \theta \beta (1 - \delta)
\]

Taking the derivative of \( g(t_1, T) \) with respect to \( T \) and setting it to zero, we have

\[
\frac{\partial g}{\partial T} = \frac{1}{T^2} \left[ \frac{\alpha}{(\theta + \beta)} e^{(\theta + \beta)t_1} - 1 \right] + \frac{1}{T} \left[ \frac{\alpha \theta \beta + (\theta + \beta)^2 (\xi_1 - \xi_3)}{(\theta + \beta)^2} e^{(\theta + \beta)t_1} \right] + \frac{\alpha \theta \beta + (\xi_2 + (\theta + \beta))}{T(\theta + \beta)} + \frac{2 \xi_4 t_1}{T} + \frac{T}{C_1 \alpha \delta (T - t_1)} - \frac{1}{T} \alpha \delta (P - C) t_1 - \frac{1}{T} (\lambda \alpha (1 - \delta)(t_1)) = 0
\]

or

\[
\left[ -\alpha (\theta + \beta)(P - C) + (\xi_1 - \xi_3)(\theta + \beta)^2 + \alpha \theta \beta \right] e^{(\theta + \beta)t_1} - \frac{\alpha \delta C_1 (\theta + \beta)^2 t_1^2 + \xi_4 (\theta + \beta)^2 t_1^2}{2} + \frac{C_1 \alpha \delta (\theta + \beta)^2 t_1^2 + \xi_2 (\theta + \beta)^2 t_1^2}{2} + \left[ \xi_2 (\theta + \beta)^2 + \alpha \theta \beta (\theta + \beta) + \alpha \delta (\theta + \beta)^2 - \lambda \alpha (1 - \delta)(\theta + \beta)^2 \right] t_1 + \alpha (\theta + \beta)(P - C) - (\xi_1 - \xi_3)(\theta + \beta)^2 - \alpha \theta \beta + C_2 (\theta + \beta)^2 = 0
\]

on simplifying, we have

\[
\phi_1 e^{(\theta + \beta)t_1} - \phi_2 t_1^2 + \phi_3 T^2 + \phi_4 t_1 + \phi_5 = 0 \tag{16}
\]
Where,
\[ \varphi_1 = -\alpha (\Theta + \beta)(P - C) + (\xi_1 - \xi_3)(\Theta + \beta)^2 + \alpha \theta P \]
\[ \varphi_2 = \frac{a \delta C_1 (\Theta + \beta)^2 + \xi_4 (\Theta + \beta)^2}{2} \]
\[ \varphi_3 = \frac{C_1 \alpha \delta (\Theta + \beta)^2}{2} \]
\[ \varphi_4 = \xi_2 (\Theta + \beta)^2 + \alpha \theta P (\Theta + \beta) + \alpha \delta (\Theta + \beta)^2 - \lambda a (1 - \delta) (\Theta + \beta)^2 \]
\[ \varphi_5 = \alpha (\Theta + \beta) (P - C) - (\xi_1 - \xi_3) (\Theta + \beta)^2 - \alpha \theta P + C_2 (\Theta + \beta)^2 \]

From eq. (16), we have
\[ T^2 = -\frac{1}{\varphi_3} (\varphi_1 (e^{(\Theta + \beta)t_1} - \varphi_2 t_1^2 + \varphi_4 t_1 + \varphi_5)) \tag{17} \]
and from eq. (15), we have
\[ T = -\frac{1}{\sigma_2} (\sigma_1 e^{(\Theta + \beta)t_1} - \sigma_3 t_1 + \sigma_4) \tag{18} \]

So that combining eqs. (17) and (18) leads to
\[ -\frac{1}{\varphi_3} (\varphi_1 (e^{(\Theta + \beta)t_1} - \varphi_2 t_1^2 + \varphi_4 t_1 + \varphi_5)) = \left[-\frac{1}{\sigma_2} (\sigma_1 e^{(\Theta + \beta)t_1} - \sigma_3 t_1 + \sigma_4)\right]^2 \]
or
\[ \varphi_1 (e^{(\Theta + \beta)t_1} - \varphi_2 t_1^2 + \varphi_4 t_1 + \varphi_5) + \frac{\varphi_2}{\sigma_2^2} (\sigma_1 e^{(\Theta + \beta)t_1} - \sigma_3 t_1 + \sigma_4) = 0 \tag{19} \]

In order to show the sufficiency condition of the objective function, for convenience we will first convert our profit function in terms of one variable \( T \) as follows:

Letting \( t_1 = \lambda \ T \), where \( 0 < \lambda < 1 \), and substituting it in eq. (12), the profit function \( g(t_1, T) \) reduces to a function of \( T \) only as

\[ g(T) = \frac{1}{T(\Theta + \beta)^2} [\lambda (\Theta + \beta) (P - C) - \alpha \theta P \frac{a \delta (\Theta + \beta)^2}{(\Theta + \beta)^2}] (e^{(\Theta + \beta)T} - 1) + \frac{\lambda}{(\Theta + \beta)^2} [a \delta (\Theta + \beta)^2 T^2 - \lambda \alpha \delta (P - C) (1 - \lambda) - \lambda \alpha (1 - \delta) (1 - \lambda)] \]

or

\[ g(T) = \frac{1}{T(\Theta + \beta)^2} \left[ \frac{\lambda (\Theta + \beta) (P - C) - \alpha \theta P \frac{a \delta (\Theta + \beta)^2}{(\Theta + \beta)^2}}{(\Theta + \beta)^2} \right] \left[ e^{(\Theta + \beta)T} - 1 \right] + \frac{\lambda}{(\Theta + \beta)^2} \left[ a \delta (P - C) (1 - \lambda) - \lambda \alpha (1 - \delta) (1 - \lambda) \right] \]

where
\[ S = \alpha(\Theta + \beta)^2(P - C) - \theta P(\Theta + \beta) - \alpha(\Theta + \beta)q - ab \] and \[ F = \frac{ab \lambda^2}{2(\Theta + \beta)} - C_1 \alpha \delta(1 - \lambda)^2 \]

**Proposition:** \( g(T) \) is concave provided \( S = \alpha(\Theta + \beta)^2(P - C) - \theta P(\Theta + \beta) - \alpha(\Theta + \beta)q - ab < 0 \)

**Proof:**

\[
\frac{dg}{dT} = -\frac{S}{T^2} \left\{ \frac{(\Theta + \beta)}{T} \right\} - \frac{(e^{(\Theta + \beta)} \lambda T)}{T^2} + \frac{2(e^{(\Theta + \beta)} \lambda T)}{T^3} - \frac{2C_2}{T^3}
\]

\[
\frac{d^2g}{dT^2} = \frac{S}{T^3} \left\{ \frac{(\Theta + \beta)^2}{T^2} (e^{(\Theta + \beta)} \lambda T - 2(\Theta + \beta) \lambda T (e^{(\Theta + \beta)} \lambda T + 2e^{(\Theta + \beta)} \lambda T)}{T^3} \right\} - \frac{2C_2}{T^3}
\]

Letting \( g(T) \) is monotonically non-decreasing for all \( x \geq 0 \). Therefore, if \( S < 0 \), then \( \frac{d^2g}{dT^2} < 0 \) which confirms that \( g(T) \) is concave. This completes the proof.

5 **Algorithm for Obtaining Optimal Solution**

The solution procedure for determining optimal solution of the inventory model is summarized in the following algorithm:

Step 1: Input the values of all parameters to obtain the value of \( S \) and if \( S < 0 \); then move to Step 2; otherwise change the values of the parameters.

Step 3: Check that the function \( h(x) > 0 \) which implies that \( g(t_1^*, T^*) \) is concave.

Step 4: If \( h(x) > 0 \) through Newton-Raphson iteration method, determine the values of \( t_1^* \) and \( T^* \) from eqs. (19) and (18) respectively.

Step 5: Compute the corresponding profit function \( g(t_1^*, T^*) \), maximum inventory level \( K \) and ordering quantity \( Q \) using eqs. (5), (10), and (12) respectively.

6 **Numerical Example**

Since it is not easy to find closed form solution to the derived model eqs. (15) and (16), we numerically find optimal solutions for given sets of model parameters using Newton Raphson’s Method. Let’s assume the model parameters to be: \( \alpha = 500 \text{units}, \beta = 0.6, \theta = 0.04, q = 0.4, b = 0.03, C_1 = 0.6 \text{unit day}, C_2 = N800 \text{/replenishment}, I = 0.8, P = N2/\text{unit}, C = N23/\text{unit}, \delta = 0.8/\text{unit} \)

Solving equation (19) using Newton-Raphson’s Method for positive \( t_1 \), we have \( t_1 = t_1^* = 6.64 \text{days} \). Putting this value in eq. (18), we get \( T = T^* = 8.97 \text{days} \). The optimum values of \( K \) and \( Q \) are \( K^* = 3938 \text{units} \) and \( Q^* = 54871 \text{units} \). Hence the profit is \( g^*(t_1^*, T^*) = N208 \text{units} \), whereas \( S = -53 \).

7 **Sensitivity Analysis**

A sensitivity analysis with respect to different associated parameters is carried out to observe the changes in the decision variables with the changes in the parameter values.
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8 Comments on the Sensitivity Analysis

1. It can be seen that as the percentage change in demand parameter $\alpha$ increases, the total depletion time $t_1$, the total cycle length $T$, the maximum inventory level $K$, the maximum order quantity $Q$ and the total profit $g(t_1, T)$ also increase which indicates that the parameter $\alpha$ is sensitive to change.

2. Percentage increase in the stock dependent demand parameter $\beta$ leads to percentage increase in total depletion time $t_1$, the total cycle length $T$, the maximum inventory level $K$, the maximum order quantity $Q$ and a very high percentage decrease in total profit $g(t_1, T)$ which shows that parameter $\beta$ is sensitive to change.

3. It can be seen that, the percentage increase in deterioration rate $\theta$ leads to percentage increase in time at which the inventory level becomes zero, total cycle length, profit maximum stock level $K$ and maximum order quantity, that is deterioration rate $\theta$ is sensitive to change. This suggests that for profit to be maximized with increase in deterioration, the maximum inventory level and the maximum order quantity should be increase as well which will also affect the total depletion period and cycle length.
4. As the $b$ increase in percentage, the total depletion period, total cycle length, maximum stock level and maximum order quantity also increase with corresponding percentage decrease in profit. This shows that parameter $b$ is sensitive to change.

5. Percentage decrease in the time dependent holding cost parameter $q$ leads to percentage decrease in total cycle length, maximum order quantity, total depletion period, maximum stock level and percentage increase in profit, while percentage increase in the time dependent demand holding cost leads to percentage increase in total cycle length, maximum order quantity and other decision variables with decrease in profit, that means parameter $q$ is sensitive to change. This signifies that decrease in time dependent holding cost will be profitable for the business.

6. Percentage increase in selling price $P$ leads to a high percentage increase in profit with about hundred percent decrease in other decision variables. This indicates that when the selling price is increase, the maximum inventory level and the maximum order quantity should be decrease by half in order to maximize profit.

7. It can be observed that when purchasing cost is either increase or decrease in percentage, the total average profit decreases. This means that increase in cost price is not advisable.

8. It can be seen that percentage decrease in shortage cost leads to percentage decrease in total depletion period, total cycle length, maximum inventory level, maximum order quantity and percentage increase in the total profit. Also, percentage increase in shortage cost leads to percentage decrease in total average profit with percentage decrease in the remaining decision variables. This indicates that shortage cost is very sensitive to change which if properly managed will be favors the business.

9. It can be observed that the ordering cost is less sensitive to change in relation to the profit and other decision variables as percentage change in the total average profit and other decision variable increase with small percentage.

10. It can be seen that percentage decrease in the backlogging parameter leads to percentage increase in total average profit and percentage decrease in other decision variables while percentage increase in backlogging parameter leads to percentage decrease in total average profit with percentage increase in the remaining decision variables. This implies that too much backlogging should not be encouraged.

11. Percentage decrease in lost sale cost leads to percentage decrease in all the decision variables with percentage increase in total average profit while percentage increase in lost sale cost leads to percentage increase in all the decision variables with percentage decrease in total average profit, which means parameter $l$ is sensitive to change. This means that much should be done to avoid lost sales.

9 Conclusions
In this paper, an inventory model is developed for partial backlogging inventory model for deteriorating item with stock dependent demand and time varying holding cost where a constant partial backlogging parameter is considered. Contrary to other related papers on partial backlogging inventory model for deteriorating item with shortages where most of them considered non instantaneous deterioration, this paper considers an instantaneous deterioration of the item under consideration because certain items such as fruits and vegetables start deteriorating as soon as they are placed on the shelf. The optimum order quantity and optimum inventory level were determined in order to maximize the total profit.
Numerical examples and sensitivity analysis have been conducted in order to show the applicability of the model to reality.

References


