MATHEMATICAL ASSOCIATION OF NIGERIA (M.A.N)

abacus

THE JOURNAL OF THE MATHEMATICAL ASSOCIATION OF NIGERIA

MATHEMATICS SCIENCE SERIES
VOLUME 49, NUMBER 4,
DECEMBER, 2022

NR-ISSN 0001 3099

Editor-in-Chief:
Professor Muhammad Lawan Kaurangini, FICA
Aliko Dangote University of Science and Technology, Wudil, Nigeria

MATHEMATICAL ASSOCIATION OF NIGERIA
(M.A.N)

abacus

THE JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF NIGERIA

MATHEMATICS SCIENCE SERIES
VOLUME 49, NUMBER 4,
DECEMBER, 2022

NR-ISSN 0001 3099

Editor-in-Chief:
Professor Muhammad Lawan Kaurangini, FICA
Kano University of Science and Technology, Wudil, Nigeria

National Executive Officers (2021-2023)

Abacus:

Submission of the paper
Authors are requested to visit www.man-nigeria.org.ng to submit manuscripts in Microsoft word and make payment accordingly.

Preparation of the manuscript
General: The manuscripts should be in English and typed with single column and single line spacing on single side of A4 paper. The first page of an article should contain;
(1) a title of paper which well reflects the contents of the paper (Arial, 12pt),
(2) all the name(s) and affiliations(s) of authors(s) (Arial, 12pt),
(3) an abstract of 100-250 words (Times New Roman, 11 pt),
(4) 3-5 keywords following the abstract, and
(5) footnote (personal title and email address of the corresponding author). The paper should be concluded by proper conclusions which reflect the findings in the paper. The normal length of the paper should be between 10 to15 journal pages.

Tables and figures: Tables and figures should be consecutively numbered and have short titles. They should be referred to in the text as following examples (e.g., Fig. 1(a), Figs. 1 and 2, Figs. 1(a)-(d) / Table 1, Tables 1-2, etc). Tables should have borders (1/2pt plane line) with the captions right before the table. Figures should be properly located in the text as an editable image file (.jpg) with captions on the lower cell.

Units and Mathematical Expressions: It is desirable that units of measurements and abbreviations should follow the System International (SI) except where the other unit system is more suitable. The numbers identifying the displayed mathematical expression should be placed in the parentheses and referred in the text as following examples (e.g., Eq. (1), Eqs. (1)-(2)). Mathematical expressions must be inserted as an object (set as Microsoft Equations 3.0) for Microsoft Word 2007 and later versions. Image-copied text or equations are not acceptable unless they are editable. The raised and lowered fonts cannot be used for superscription and subscription.

References : A list of references which reflect the current edition of APA format, to locate after conclusion of the paper.

Review
All the submitted papers will undergo a peer-review process, and those papers positively recommended by at least two expert reviewers will be finally accepted for publication in the Abacus Journals or after any required modifications are made and a payment of **₦15,000.00** as publication fee. The normal length of the paper should be between 8 to 12 pages. Any extra page will attract additional charges at the rate of **₦500.00** per page.

Copyright
Submission of an article to Abacus Journal implies that it presents the original and unpublished work, and not under consideration for publication elsewhere. On acceptance of the submitted manuscript, it is implied that the copyright thereof is transferred to the Abacus. The Transfer of Copyright Agreement may also be submitted.

Editorial Board

Associate Editors (2021-2023)

TABLE OF CONTENTS

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

STABILITY ANALYSIS OF SOLUTIONS FOR A CLASS OF THIRD-ORDER NON LINEAR DUFFING-TYPE DIFFERENTIAL EQUATION

Kalu, Uchenna[1] and Anozie, V.O[2]
Department of Mathematics,
Michael Okpara University of Agriculture, Umudike Abia State, Nigeria.
Email: uchennakalu304@gmail.com[1], anozievictorobinna@gmail.com[2]

Abstract

In this paper, the eigenvalue method is applied to study the stability of solutions for a class of third-order nonlinear Duffing-type differential equation. By dimensionalizing the equation to a first-order system, the nonlinear parts of each of the equivalent system derived is linearized using Maclaurin series expansion, stability solutions were investigated and it was concluded that since all the eigenvalues do not all have negative real parts, the system is unstable.

Keywords: Duffing-type ODE, eigenvalue method, Linearization, Stability.

Introduction

Several articles have appeared in the literature relating to the stability of periodic solutions of nonlinear differential equations of Duffing-type [1, 2, 3, and 4]. The existence of solutions of ordinary differential equation using implicit function theorem have been investigated by several notable authors [3, 4, and 5]. Other stability analysis of fractional Duffing oscillator used implicit function theorem to show the existence of periodic solutions for nonlinear partial differential equations [3, 7]. Other researchers investigated the stability of solutions of certain order types of delay differential equations [9-12], etc.

Consider the Duffing type equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx + dx^2 + 2x^3 = g(t)$$

(1.1)

where a, b, c are real constants and $g(t)$ is continuous, this has been used in engineering, economics, physics and many other physical phenomena. Given its characteristic oscillatory and chaotic nature many scientists are inspired by this nonlinear differential equation given its ability to replicate similar dynamics in our natural world. The nonlinear differential equation is used to model damped and derived oscillators [6]. These equations together with the Vander Pol's equation have become one of the most common examples in nonlinear oscillations. Due to the importance of the Duffing equation in real world problems, the study of existence of solution of the equation has continued to attract the attention of many researchers. The existence of solution of Duffing equation of the general form:

$$\ddot{x} + c\dot{x} + g(t, x) = g(t) \tag{1.2}$$

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

where $g(t)$ is continuous and 2π-periodic in $t \in \mathbb{R}$, $g(t,x) = ax + bx^2 + \mathcal{B}x^3$ and $\mathcal{B} = 2$ represents a hard spring, carried out by [1].

The aim of this paper is to investigate the stability of solutions for a class of Third Order Nonlinear Differential equation of the Duffing-type

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx + dx^2 + 2x^3 = f(t)$$

(1.3)

where a, b, c, d are real constants and $f(t)$ is continuous using the eigenvalue method. The eigenvalues $\lambda_1, \dots$ are obviously the roots of the polynomial.

Preliminaries

In order to reach our main results we will first give some important stability criteria for the general autonomous delay differential system.

Definition 2.1 (Stability)

Consider the nonlinear time-invariant system

$$\dot{x} = f(x), \quad \dot{f}: \mathbb{R} \to \mathbb{R}, \tag{2.2}$$

A point $x_e \in \mathbb{R}$, is an equilibrium point of the system if $f(x) = 0$.

We remark that x_e is an equilibrium point if and only if $x(t) = x_e$ is a trajectory.

Suppose x_e is an equilibrium point, then

i. The system is globally asymptotically stable if for every trajectory x, we have $x(t) \to x_e$ as t$\to \infty$.

ii. The system is locally asymptotically stable near or at x_e if there is an $\mathcal{R} > 0 \ni \| x(0) - x_e \| \leq \mathcal{R} \quad \Rightarrow x(t) \longrightarrow x_e \; as \; t \longrightarrow \infty$.

Definition 2.2

Consider also the linear system

$$\dot{x} = Ax$$

2.4

i. The system is said to be globally asymptotically stable with $\varkappa = 0$ if and only if $\Re_e \lambda_i(A) < 0, i = 1,2,3, \dots n$, where $\Re_e$ *means the real part.*

ii. The system is said to be locally asymptotically stable (near $\varkappa = 0$) if and only if $\Re_e \lambda_i(A) < 0$,
$i = 1,2,3, \dots n. Thus\ for\ linear\ system\ , Locally\ asymptoticaally\ stable \Leftrightarrow Globally\ asymptotically\ stable$).

Definition 2.3 (Asymptotic stability)

The equilibrium point $\varkappa$ is said to be asymptotically stable, if for all $\varepsilon > 0, \exists\ \delta > 0\ \ni$

i. $f(t; 0, \bar{x}) \in \beta(\varkappa, \varepsilon)$ for all$t \geq 0$.

ii. $\lim_{t\to\infty} f(t, 0, \bar{x}) = \varkappa$

Definition 2.4 (Lyapunov function)

Consider the differential equation

$$\dot{x} = f(x), \qquad f(0) = 0$$

Where the solutions are unique and vary continuously with the initial data. Let $V: \mathbb{R}^n \to \mathbb{R}$ be continuous together with its first partial derivatives, $\frac{\partial v}{\partial x}$ $(i = 1,2, \dots)$ on some open set $\Omega \subset \mathbb{R}^n$, where $\Omega = B_r(0)$, $\Omega = \{x: x\epsilon\mathbb{R}^n: \parallel x \parallel < r, where\ r\ is\ the\ radius\ of\ the\ ball\}$

i. A Function $V: \Omega \to \mathbb{R}$ is said to be positive definite/negative definite if $v(0) = 0\ and\ v$ assumes positive/negative values on Ω.

ii. A Function $V: \Omega \to \mathbb{R}$ is said to be positive/negative semi definite if $v(0) = (0)\ and\ v(x) \geq 0$ or $v(x) \leq 0$ on Ω.

If the function assumes arbitrary values, then it is said to be indefinite.

Theorem 1

The critical point $0 \in \mathbb{R}^n$ for the linear system $\dot{x} = A\underline{x}$ is asymptotically stable provided that all the eigenvalues of A have negative real parts otherwise it is unstable.

Theorem 2

Consider $\dot{x} = f(x)$, and assume that $f(0) = 0$

Linearization

$$\dot{x} = A(\underline{x}) + g(\underline{x}), \quad \parallel g(x) \parallel = 0(\parallel \underline{x} \parallel)\ as\ x \to 0$$

i. $\Re_e \lambda_k(A) < 0$, for $k \Longrightarrow x = 0\ is$ Locally Asymptotically Stable $(L.A.S)$.

ii. $\exists\ K: \Re_e \lambda_k(A) > 0 \Longrightarrow x = 0\ is\ unstable.$

Definition 2.5 (Higher Order ODE and Reduction to the first Order System)

A general ODE of the order n resolved with respect to the highest derivative can be written in the form:

$$y^{(n)} = F(t, y, y', \dots, y^{(n-1)}) \tag{2.1}$$

Where t is an independent variable and $y(t)$ is an unknown function. It is sometimes more convenient to replace this equation by a system of ODEs of the first order. And any scalar

ODE can be represented by any system of first order differential equation. In practice, the system form is widely used in problems connected with stability, boundedness and periodicity of solutions.

Let $x(t)$ be a vector function of a real variable t, which takes values in $\mathbb{R}^n$. Denote by x_k the components of x, then the derivative $x'(t)$ is defined component-wise by $x'(x_1', x_2', \dots, x_n')$.

Now, consider a vector ODE of the first order

$$x' = f(t, x) \tag{2.2}$$

Where f is a given function of $n + 1$ variables, which takes values in $\mathbb{R}^n$, that is $f: \Omega \to \mathbb{R}^n$, where Ω is an open subset of $\mathbb{R}^{n+1}$. Denoting by f_k the components of f, we can rewrite the vector equation (2.2) as a system of n scalar equations:

$$\left.\begin{aligned} x_1' &= f_1(t, x_1, \dots, x_n) \\ &\dots \\ x_k' &= f_1(t, x_1, \dots, x_n) \\ &\dots \\ x_n' &= f_1(t, x_1, \dots, x_n) \end{aligned}\right\} \tag{2.3}$$

A system of ODEs of the form (2.3) is called the normal system. This can be shown how (2.1) can be reduced to (2.3).

Suppose we associate the vector function $x = \left(y, y', \dots, y^{(n-1)}\right)$, which takes in $\mathbb{R}^n$. That is

$$x_1 = y, x_2 = y', \dots, x_n = y^{(n-1)} \Rightarrow x' = \left(y', y'', \dots, y^{(n)}\right) \tag{2.4}$$

And using (2.1), we obtain a system of equations

$$\left.\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\dots \\ x_{n-1}' &= x_n \\ x_n' &= f(t, x_1, \dots, x_n) \end{aligned}\right\} \tag{2.5}$$

Main Results

Now consider the Duffing-type equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx + dx^2 + 2x^3 = g(t)$$

(3.1)

Let $g(t) = 0$, then equation (3.1) becomes

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx + dx^2 + 2x^3 = 0$$

(3.2)

We obtain the first order systems from the scalar differential equation (3.2) by letting

$$\left.\begin{matrix} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = w \end{matrix}\right\} \quad (3.3)$$

The equation (3.2) becomes

$$\left.\begin{matrix} \ddot{y} + a\dot{y} + by + cx + dx^2 + 2x^3 = 0 \\ \dot{z} + az + by + cx + dx^2 + 2x^3 = 0 \\ \dot{z} = -cx - by - az - dx^2 - 2x^3 \end{matrix}\right\} \quad (3.4)$$

The equivalent system is now

$$\left.\begin{matrix} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -cx - by - az - dx^2 - 2x^3 \end{matrix}\right\}$$

(3.5)

The equation (3.5) is the first of the equivalent system obtained directly from the scalar equation. This can be written in matrix form as;

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -dx^2 - 2x^3 \end{pmatrix} \quad \Rightarrow \quad \dot{z} = A\underline{z} + g(\underline{z})$$

where the matrix,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{pmatrix}, \quad \underline{\dot{z}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \underline{z} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } g(\underline{z}) = \begin{pmatrix} 0 \\ 0 \\ -dx^2 - 2x^3 \end{pmatrix}$$

The equation (3.2) can be written as;

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx + dx^2 + 2x^3 \equiv \frac{d}{dt}(\ddot{x} + a\dot{x}) + b\dot{x} + cx + dx^2 + 2x^3 = 0$$

(3.6)

Let $\ddot{x} + a\dot{x} = z$

$$\dot{y} + ay = z \Rightarrow \dot{y} = -ay + z$$

Also (3.6) can be written as;

$$\left.\begin{aligned} \frac{d}{dx}(\ddot{x} + a\dot{x}) + b\dot{x} + cx + dx^2 + 2x^3 \equiv \frac{d}{dt}(z) + b\dot{x} + cx + dx^2 + 2x^3 = 0 \\ \dot{z} + by + cx + dx^2 + 2x^3 = 0 \\ \dot{z} = -by - cx - dx^2 - 2x^3 \end{aligned}\right\}$$

(3.7)

The equivalent system is now

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -ay + z \\ \dot{z} &= -cx - by - dx^2 - 2x^3 \end{aligned}\right\}$$

(3.8)

The equation (3.8) is the second of the three equivalent first order systems which can be written in matrix form as;

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -a & 1 \\ -c & -b & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -dx^2 - 2x^3 \end{pmatrix} \Rightarrow \dot{\underline{z}} = A\underline{z} + g(\underline{z})$$

where,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -a & 1 \\ -c & -b & 0 \end{pmatrix}, \ \dot{\underline{z}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \underline{z} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } g(\underline{z}) = \begin{pmatrix} 0 \\ 0 \\ -dx^2 - 2x^3 \end{pmatrix}$$

The equation (3.2) can be written as;

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx + dx^2 + 2x^3 \equiv \dddot{x} + \frac{d}{dt}(a\dot{x} + bx) + cx + dx^2 + 2x^3 = 0$$

(3.9)

Let $a\dot{x} + bx = y$

$ay + bx = \dot{x} \Longrightarrow \dot{x} = bx + ay$

Also equation (3.9) can be written as;

$$\left.\begin{aligned} \dddot{x} + \frac{d}{dt}(a\dot{x} + bx) + cx + dx^2 + 2x^3 \equiv \dddot{x} + \frac{d}{dt}(y) + cx + dx^2 + 2x^3 = 0 \\ \ddot{y} + \dot{y} + cx + dx^2 + 2x^3 = 0 \\ \dot{z} + z + cx + dx^2 + 2x^3 = 0 \\ \dot{z} = -cx - z - dx^2 - 2x^3 \end{aligned}\right\}$$

(3.10)

The equivalent system is now

$$\left.\begin{aligned} \dot{x} &= bx + ay \\ \dot{y} &= z \\ \dot{z} &= -cx - z - dx^2 - 2x^3 \end{aligned}\right\}$$

(3.11)

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

Equation (3.11) is the third of the equivalent first order system which can be written in a matrix form as;

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} b & a & 0 \\ 0 & 0 & 1 \\ -c & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -dx^2 - 2x^3 \end{pmatrix} \quad \Rightarrow \quad \dot{z} = A\underline{z} + g(\underline{z})$$

where,

$$A = \begin{pmatrix} b & a & 0 \\ 0 & 0 & 1 \\ -c & 0 & -1 \end{pmatrix}, \quad \underline{\dot{z}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \underline{z} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } g(\underline{z}) = \begin{pmatrix} 0 \\ 0 \\ -dx^2 - 2x^3 \end{pmatrix}$$

Stability Analysis

We linearize the nonlinear parts of each of the equivalent system derived using Maclaurin series expansion i.e.

$$g(\underline{x}) = g(\underline{0}) + \underline{x}g'(\underline{0}) + \frac{1}{2!} \parallel \underline{x} \parallel^2 g''(\underline{0}) + \cdots, \qquad \parallel g(\underline{x}) \parallel = 0 \; as \; x \to 0$$

The linearized term is now

$$g(\underline{x}) = g(\underline{0}) + \underline{x}g'(\underline{0})$$

But $g(\underline{0}) = 0$

$$\Rightarrow g(\underline{x}) = \underline{x}g'(\underline{0})$$

Hence, the system $\dot{z} = A\underline{z} + g(\underline{z})$ becomes

$$\dot{x} = A\underline{x} + \underline{x}g'(\underline{0}) \qquad 4.1$$

From the above we see that $g'(\underline{0})$ is necessarily an $n \times n$ matrix since it must be compatible with matrix A. Then we have a linearized system given by $\dot{x} = Bx$, where B is called the linearized matrix i.e $B = A + g'(\underline{0})$.

Applying this we have that

$$g(\underline{x}) = \begin{pmatrix} 0 \\ 0 \\ -dx^2 - 2x^3 \end{pmatrix} \quad \Rightarrow \quad g'(\underline{0}) = \begin{pmatrix} 0 \\ 0 \\ -2d(0) - 6(0)^2 \end{pmatrix} = 0$$

Hence, $\dot{z} = Az$

Now, we want to test the stability for each of the matrices A derived using eigenvalue method i.e. $|A_n - \lambda I| = 0; n = 1, 2, 3.$

Where, $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -c & -d & -a \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -a & 1 \\ -c & -b & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} b & a & 0 \\ 0 & 0 & 1 \\ -c & 0 & -1 \end{pmatrix}$

For $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{pmatrix}$,

$$\Rightarrow |A_1 - \lambda I| = \left|\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -c & -b & -a-\lambda \end{vmatrix} = 0$$

$$-\lambda \begin{vmatrix} -\lambda & 1 \\ -b & -a-\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ -c & -a-\lambda \end{vmatrix} = 0 \qquad \Rightarrow -\lambda[-\lambda(-a\lambda) + b] -$$

$(0 + c) = 0$

$$-\lambda^2 a - \lambda^3 - \lambda b + c = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 a + \lambda b + c = 0 \qquad 4.2$$

We have one condition on three constants, two of which are therefore a free choice, choose $a = 1 \text{ and } b = c = -1$ for convenience.

Equation (4.2) becomes

$$\lambda^3 + \lambda^2 - \lambda - 1 = 0 \qquad 4.3$$

$$\Rightarrow \quad \lambda = \begin{pmatrix} +1 \\ -1 \\ -1 \end{pmatrix}$$

Since all the eigenvalues are not negative real parts then, equation (4.3) is unstable.

For $A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -a & 1 \\ -c & -b & 0 \end{pmatrix}$,

$$\Rightarrow |A_2 - \lambda I| = \left|\begin{pmatrix} 0 & 1 & 0 \\ 0 & -a & 1 \\ -c & -b & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -a-\lambda & 1 \\ -c & -b & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -\lambda \begin{vmatrix} -a-\lambda & 1 \\ -b & -\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 0 & -\lambda \end{vmatrix} = 0 \qquad \Rightarrow -\lambda[-\lambda(-a-\lambda) + b] = 0$$

Putting $a = 1 \text{ and } b = c = -1$ gives

$$\lambda^3 + \lambda^2 - \lambda = 0$$

4.4

$$\Rightarrow \quad \lambda = \begin{pmatrix} 0 \\ 1.62 \\ -0.62 \end{pmatrix}$$

Since all the eigenvalues are not negative real parts, the equation (4.4) is unstable

For $\qquad A_3 = \begin{pmatrix} b & a & 0 \\ 0 & 0 & 1 \\ -c & 0 & -1 \end{pmatrix}$,

$$\Rightarrow |A_3 - \lambda I| = \left|\begin{pmatrix} b & a & 0 \\ 0 & 0 & 1 \\ -c & 0 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right| = \begin{vmatrix} b-\lambda & a & 0 \\ 0 & -\lambda & 1 \\ -c & 0 & -1-\lambda \end{vmatrix} = 0$$

$$(b-\lambda) \begin{vmatrix} -\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} - a \begin{vmatrix} 0 & 1 \\ -c & -1-\lambda \end{vmatrix} = 0 \qquad \Rightarrow (b-\lambda)(\lambda + \lambda^2) = 0$$

Putting $a = 1$ *and* $b = c = -1$ gives

$$\lambda^3 + 2\lambda^2 + \lambda - 1 = 0 \qquad 4.5$$

$$\Rightarrow \quad \lambda = \begin{pmatrix} 0.46557 \\ -1.232786 + 0.79285 \\ -1.232786 - 0.79285 \end{pmatrix}$$

Equation (4.5) is also unstable.

Conclusion

The eigenvalue method is very easy to handle as verified in this paper. The need to first convert higher order differential equations to first order differential equations has been a key to easily solving higher order differential equations. Therefore, it is recommended that one can check for the stability of the system by picking only one of the equivalent first order systems derived instead of checking for each of the equivalent system. Furthermore, this method tends to be complex if higher numbers are assumed hence; the technique requires the smallest possible number to be assumed. However, since all the eigenvalues of the matrices do not all have negative real parts, we conclude that the Duffing system (3.1) is unstable.

References

Osisiogu, U.A., Eze, E.O. and Obasi, U.E. (2016): Existence and stability of periodic solutions for a class of second order nonlinear differential equations. *Journal of the Nigerian Association of Mathematical Physics*, Vol. 36, No.2, pp 37-41, 43-48.

Kernal, A. (2001): Stability of solution of linear differential equations with periodic coefficient, *Sib. Mat. J.* 41, Number 6, 1005-1010.

Kreici, P. (1974): Hard implicit function theorem and small periodic solutions to partial differential equations. *Commentationes Mathematical Universitatis Carolinae* Vol. 25, No. 3, 519536.

Deepmala, H. and Pathak, K. (2013): A study on some problems on existence of solutions for nonlinear functional-integral equation. *Acta Mathematica Scientia*, 33B(5),pp 1305-1313.

Ueda, Y. (1979): Randomly transitional phenomena in the system, governed by Duffing's equation. *Journal of Statistical Physics*, 20(2), pp. 181-196.

Oyesanya, M.O. and Nwamba, J.I. (2013): Stability analysis of damped cubic-quintic Duffing oscillator. *World Journal of Mechanics* 3: 43-57.

Oyesanya, M.O. and Ejikeme, C.L. (2016). Stability analysis of fractional Duffing oscillator II. *Transactions of the Nigerian Association of Mathematical Physics*. 2, 343-352.

Lakshmanan, S., Rihan, F. A., Rakkiyappan, R. and Park, J. H. (2014): Stability analysis of the differential genetic regulatory networks model with time-varying delays and Markovian jumping parameters, *Nonlinear Analysis: Hybrid Systems*, vol. 14, pp. 1–15.

Sinha, A.S.C. On stability of some third and fourth order delay-differential equations. *Journal of Information and Control*. 23; 165-172.

Sadek, A. I. (2004): On the stability of solutions of certain fourth order delaydifferential equations. *Applied Mathematics and Computation*; 148(2):587–597.

Tunc, C. (2005): Some stability results for the solutions of certain fourth order delay differential equation. *Journal of Differential Equations and Applications*. 4: 165-174.

Bereketoglu, H. (1998). Asymptotic stability in a fourth order delay differential equation. *Journal of Dynam systems Application*. 7(1); 105-115.

SOME TRIPLED FIXED POINT RESULTS IN C*-ALGEBRA b-CAUCHY SPACES

Aniki, S. A.
Department of Mathematics, Faculty of Science, Confluence University of Science and Technology, Osara, Kogi State.
anikisa@custech.edu.ng

Abstract
Tripled fixed point theory has significantly proven its relevance over coupled fixed point in diverse areas of mathematical analysis, and its applications to problem solving. This work is predicated on the concept of C^*-algebra-valued b-metric space, and it establish fixed point result of the form T:X^3→X, which satisfies some new contractive conditions. The outcome of this study is an extension to some results in recent literature.

Keywords: fixed point, tripled fixed point, C^*-algebra-valued, b-metric spaces, Cauchy spaces.

Introduction

Fixed point is of great interest in Mathematics as it is in numerous fields of applied sciences. Tripled fixed point is an extension to coupled fixed point theorem. Bahkitin (1989) initiated $b-$metric spaces as a deduction of metric space. Since then, several other deductions to $b-$metric spaces (Czerwick, 1993) and quasi $-b-$ metric spaces (Czerwick, 1998; Kirk and Shahzad, 2014) were also introduced.

Ma and Jiang (2014) initiated the idea of a C^*-algebra which deduced the notion of $b-$ metric, and demonstrated some fixed point results for self-map in the setting of certain new conditions. Aydi et al. (2015) also worked on C^*-algebra metric and deduced the Banach contraction on the spaces.

As a motivation to our results, we obtain tripled fixed point which satisfies some contractive conditions based on the concept of C^* algebra-valued b^*-metric space as in (Kamran et el., 2016; Bai, 2016).

1. Methodology

We recount some preliminaries of C^*-algebra which can be found in (Davidson, 1996; Kuman et al., 2013; Ma et al., 2014; Bai, 2016; Kamran et al., 2016).

Definition 1 An involution on an algebra $\mathbb{A}$ is a conjugate linear map $a \mapsto a^*$, that is $(a^*)^* = a$ and $(ab)^* = a^*b^*$ for $a, b \in \mathbb{A}$. The pair $(\mathbb{A}, *)$ is called a $*$-algebra. If $\mathbb{A}$ contains $1_{\mathbb{A}}$, an identity, then $(\mathbb{A}, *)$ is called a united $*$-algebra. A $*$-algebra $\mathbb{A}$ with a complete submultiplicative norm $\|a^*\| = \|a\|$ is a Banach $*$-algebra, which implies that $\mathbb{A}$ is known as C^*-algebra.

Definition 2 Let $\mathbb{A}$ be a C^*-algebra and X be non-empty. Then, $\mathbb{A}'_+$ is such that $\|b\| \geq 1$ and $d_b: X^2 \to \mathbb{A}_+$ is a C^*-algebra-valued b-metric on X if the following conditions are meet for $x, y, z \in \mathbb{A}$:

C1. $d_b(x, y) = 0_{\mathbb{A}}$ if and only if $x = y$

C2. $d_b(x,y) = d_b(y,x)$;
C2. $d_b(x,y) \preccurlyeq b[d_b(x,z) + d_b(z,y)]$
Then $(X, \mathbb{A}, d_b)$ is a C^*-algebra b-metric space with coefficient b.
Definition 3 Let $(X, \mathbb{A}, d_b)$ be a C^*-algebra-valued b-metric, $x \in X$ and $\{x_n\}$ in X. Then:

1. $\{x_n\}$ converges to x with respect to $\mathbb{A}$ for any $\varepsilon > 0$, there exist an $N \in \mathbb{N}$ such that $\|d_b(x_n, x)\| < \varepsilon$ for all $n > N$
2. $\{x_n\}$ is Cauchy if $\varepsilon > 0$, for $N \in \mathbb{N}$ such that $\|d_b(x_n, x_m)\| < \varepsilon$ for all $m, n > N$
3. $(X, \mathbb{A}, d_b)$ is complete if every Cauchy sequence in X is convergent with respect to $\mathbb{A}$.

Lemma 1 Let $\mathbb{A}$ be a unital C^*-algebra with a unit $1_{\mathbb{A}}$.

1. For any $x \in \mathbb{A}_+$, $x \preccurlyeq 1_{\mathbb{A}} \Leftrightarrow \|x\| \leq 1$;
2. If $a \in \mathbb{A}_{+,}$ with $\|a\| \leq \frac{1}{2}$, then $1_{\mathbb{A}} - a$ is inevitable and $\|a(1_{\mathbb{A},} - a)^{-1}\| < 1$;
3. Assume that $a, b \in \mathbb{A}$ with $a, b \succcurlyeq 0_{\mathbb{A},}$ and $ab = ba$, then $ab \succcurlyeq 0_{\mathbb{A},}$;
4. Let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$ with $b \succcurlyeq c \succcurlyeq 0_{\mathbb{A}}$, and $1_{\mathbb{A}} - a \in \mathbb{A}'_+$ is an invertible map, then
$$(1_{\mathbb{A},} - a)^{-1} b \succcurlyeq (1_{\mathbb{A}} - a)^{-1} c;$$
5. If $a, b, c \in \mathbb{A}_h = \{x \in \mathbb{A}: x = x^*\}$ and $a \in \mathbb{A}$, then $b \preccurlyeq c \Longrightarrow a^* b a \preccurlyeq a^* c a$;
6. If $0_{\mathbb{A}} \preccurlyeq a \preccurlyeq b$, then $\|a\| \leq \|b\|$.

Lemma 2 The sum of two elements which are both positive in a C^*-algebra is a positive element.

Definition 4 Let $(X, \mathbb{A}, d_b)$ be a C^*-algebra-valued b-metric space. Then, $(x,y) \in X^2$ is a coupled fixed point of $T: X^2 \to X$ if $T(x,y) = x$ and $T(y,x) = y$.

Definition 5 Let $(X, \mathbb{A}, d_b)$ be a C^*-algebra b-metric space. An element $(x,y,z) \in X^3$ is a tripled fixed point of $T: X^3 \to X$ if $T(x,y,z) = x$, $T(y,x,z) = y$, and $T(z,y,x) = x$.

2. Main Results

We show tripled fixed point result for certain contractive conditions for C^*-algebra-valued b-metric space.

Theorem 1 Let $(X, \mathbb{A}, d_b)$ be a C^*-algebra-valued b-metric space. Let $T: X^3 \longrightarrow X$ satisfy the following conditions:

$$d_b(T(u,v,w), T(p,q,r)) \preccurlyeq a^* d_b(u,p) a + a^* d_b(v,q) a + a^* d_b(w,r) a,$$
$$\forall\ u, v, w, p, q, r \epsilon X, \qquad (1)$$

where $a \in \mathbb{A}$ with $3\|a\|^2\|b\| < 1$. Then, T has a unique tripled fixed point in X. Moreover, T encompasses a unique fixed point in X.

Proof

Let $u_0, v_0, w_0 \in X$. Define three sequences $\{u_0\}, \{v_0\}$ and $\{w_0\}$ in X by the iteration procedure as

$$u_{n+1} = T(u_n, v_n, w_n), \qquad v_{n+1} = T(v_n, u_n, w_n) \ and \ w_{n+1} = T(w_n, v_n, u_n)$$

By the utilization of condition (1), for $n \in \mathbb{N}$, we obtain

$$\begin{aligned} &d_b(u_n, u_{n+1}) = d_b\big(T(u_{n-1}, v_{n-1}, w_{n-1}), T(u_n, v_n, w_n)\big) \\ &\preccurlyeq a^* d_b(u_{n-1}, u_n)a + a^* d_b(v_{n-1}, v_n)a + a^* d_b(w_{n-1}, w_n)a \\ &= a^* M_n a, \end{aligned} \tag{2}$$

where

$$M_n = d_b(u_{n-1}, u_n) + d_b(v_{n-1}, v_n) + d_b(w_{n-1}, w_n) \tag{3}$$

Similarly, we get

$$\begin{aligned} d_b(v_n, v_{n+1}) &= d_b\big(T(v_{n-1}, u_{n-1}, w_{n-1}), T(v_n, u_n, w_n)\big) \\ &\preccurlyeq a^* d_b(v_{n-1}, v_n)a + a^* d_b(u_{n-1}, u_n)a + a^* d_b(w_{n-1}, w_n)a \\ &= a^* M_n a, \end{aligned} \tag{4}$$

and

$$d_b(w_n, w_{n+1}) = d_b\big(T(w_{n-1}, v_{n-1}, u_{n-1}), T(w_n, v_n, u_n)\big) = a^* M_n a, \tag{5}$$

From (1)-(5) we have,

$$\begin{aligned} M_{n+1} = d_b(u_n, u_{n+1}) &+ d_b(v_n, v_{n+1}) + d_b(w_n, w_{n+1}) \\ &\preccurlyeq a^*[d_b(u_{n-1}, u_n) + d_b(v_{n-1}, v_n) + d_b(w_{n-1}, w_n)]a \\ &+ a^*[d_b(v_{n-1}, v_n) + d_b(u_{n-1}, u_n) + d_b(w_{n-1}, w_n)]a \\ &+ a^*[d_b(w_{n-1}, w_n) + d_b(v_{n-1}, v_n) + d_b(u_{n-1}, u_n)]a \\ &\preccurlyeq 3a^*[d_b(u_{n-1}, u_n) + d_b(v_{n-1}, v_n) + d_b(w_{n-1}, w_n)]a \\ &\preccurlyeq \left(\sqrt{3}a\right)^* M_n\left(\sqrt{3}a\right) \end{aligned} \tag{6}$$

Thus, from (6) and the conditions of Lemma 1, we obtain

$$0_{\mathbb{A}} \preccurlyeq M_{n+1} \preccurlyeq \left(\sqrt{3}a\right)^* M_n\left(\sqrt{3}a\right) \preccurlyeq \cdots \preccurlyeq \left[\left(\sqrt{3}a\right)^*\right]^n M_1 \left(\sqrt{3}a\right)^n$$

If $M_1 = 0_{\mathbb{A}}$, Definition 4 shows that (u_0, v_0, w_0) is a tripled fixed point of T. Then, for $m, n \in \mathbb{N}$ with $m > n$, $0_{\mathbb{A}} \preccurlyeq M_1$ and Definition 2 entails

$$\begin{aligned} d_b(u_n, u_m) &\preccurlyeq b[d_b(u_n, u_{n+1}) + d_b(u_{n+1}, u_m)] \\ &\preccurlyeq b d_b(u_n, u_{n+1}) + b^2[d_b(u_{n+1}, u_{n+2}) + d_b(u_{n+2}, u_m)] \end{aligned}$$

$$= bd_b(u_n, u_{n+1}) + b^2 d_b(u_{n+1}, u_{n+2}) + b^2 d_b(u_{n+2}, u_m)$$

$$\preccurlyeq bd_b(u_n, u_{n+1}) + b^2 d_b(u_{n+1}, u_{n+2}) + \cdots + b^{m-n-1} d_b(u_{m-2}, u_{m-1}) + b^{m-n-1} d_b(u_{m-1}, u_m).$$

Similarly, we have

$$d_b(v_n, v_m) \preccurlyeq bd_b(v_n, v_{n+1}) + b^2 d_b(v_{n+1}, v_{n+2}) + \cdots + b^{m-n-1} d_b(v_{m-2}, v_{m-1}) + b^{m-n-1} d_b(v_{m-1}, v_m).$$

and

$$d_b(w_n, w_m) \preccurlyeq bd_b(w_n, w_{n+1}) + b^2 d_b(w_{n+1}, w_{n+2}) + \cdots + b^{m-n-1} d_b(w_{m-2}, w_{m-1}) + b^{m-n-1} d_b(w_{m-1}, w_m).$$

Hence,

$$\begin{aligned} d_b(u_n, u_m) + d_b(v_n, v_m) + d_b(w_n, w_m) &= b[d_b(u_n, u_{n+1}) + d_b(v_n, v_{n+1}) + d_b(w_n, w_{n+1})] \\ &+ b^2[d_b(u_{n+1}, u_{n+2}) + d_b(v_{n+1}, v_{n+2}) + d_b(w_{n+1}, w_{n+2})] + \cdots \\ &+ b^{m-n-1}[d_b(u_{m-2}, u_{m-1}) + d_b(v_{m-2}, v_{m-1}) + d_b(w_{m-2}, w_{m-1})] \\ &+ b^{m-n-1}[d_b(u_{m-1}, u_m) + d_b(v_{m-1}, v_m) + d_b(w_{m-1}, w_m)]. \end{aligned}$$

$$\preccurlyeq bM_{n+1} + b^2 M_{n+2} + \cdots + b^{m-n-1} M_{m-1} + b^{m-n-1} M_m$$

$$\preccurlyeq b(\sqrt{3}a)^* M_n(\sqrt{3}a) + b^2(\sqrt{3}a)^* M_{n+1}(\sqrt{3}a) + \cdots + b^{m-n-1}(\sqrt{3}a)^* M_{n-2}(\sqrt{3}a) + b^{m-n-1}(\sqrt{3}a)^* M_{n-1}(\sqrt{3}a)$$

$$= b\left[(\sqrt{3}a)^*\right]^n M_1 (\sqrt{3}a)^n + b^2\left[(\sqrt{3}a)^*\right]^{n+1} M_1 (\sqrt{3}a)^{n+1} + \cdots + b^{m-n-1}\left[(\sqrt{3}a)^*\right]^{m-2} M_1(\sqrt{3}a)^{m-2} + b^{m-n-1}\left[(\sqrt{3}a)^*\right]^{m-1} M_1 (\sqrt{3}a)^{m-1}$$

$$= b\sum_{i=n}^{m-2} b^{i-n}\left[(\sqrt{3}a)^*\right]^i M_1 (\sqrt{3}a)^i + b^{m-n-1}\left[(\sqrt{3}a)^*\right]^{m-1} M_1 (\sqrt{3}a)^{m-1}$$

$$= b\sum_{i=n}^{m-2} b^{i-n}\left[(\sqrt{3}a)^*\right]^i M_1^{\frac{1}{2}} M_1^{\frac{1}{2}}(\sqrt{3}a)^i + b^{m-n-1}\left[(\sqrt{3}a)^*\right]^{m-1} M_1^{\frac{1}{2}} M_1^{\frac{1}{2}} (\sqrt{3}a)^{m-1}$$

$$= b\sum_{i=n}^{m-2} b^{i-n}\left(M_1^{\frac{1}{2}}(\sqrt{3}a)^i\right)^*\left(M_1^{\frac{1}{2}}(\sqrt{3}a)^i\right) + b^{m-n-1}\left(M_1^{\frac{1}{2}}(\sqrt{3}a)^{m-1}\right)^*\left(M_1^{\frac{1}{2}}(\sqrt{3}a)^{m-1}\right)$$

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

$$= b\sum_{i=n}^{m-2} b^{i-n}\left|M_1^{\frac{1}{2}}(\sqrt{3}a)^i\right|^2 + b^{m-n-1}\left|M_1^{\frac{1}{2}}(\sqrt{3}a)^{m-1}\right|^2$$

$$\preccurlyeq \left\|b\sum_{i=n}^{m-2} b^{i-n}\left|M_1^{\frac{1}{2}}(\sqrt{3}a)^i\right|^2\right\| 1_{\mathbb{A}} + \left\|b^{m-n-1}\left|M_1^{\frac{1}{2}}(\sqrt{3}a)^{m-1}\right|^2\right\| 1_{\mathbb{A}}$$

$$\preccurlyeq \|b\|\sum_{i=n}^{m-2} \|b^{i-n}\| \left\|M_1^{\frac{1}{2}}\right\|^2 \left\|(\sqrt{3}a)^i\right\|^2 1_{\mathbb{A}} + \|b^{m-n-1}\| \left\|M_1^{\frac{1}{2}}\right\|^2 \left\|(\sqrt{3}a)^{m-1}\right\|^2 1_{\mathbb{A}}$$

$$\preccurlyeq \|b\|^{1-n} \left\|M_1^{\frac{1}{2}}\right\|^2 \sum_{i=n}^{m-2} \|b\|^i \left\|(\sqrt{3}a)^2\right\|^i 1_{\mathbb{A}} + \|b\|^{-n} \left\|M_1^{\frac{1}{2}}\right\|^2 \|b\|^{m-1} \left\|(\sqrt{3}a)^2\right\|^{m-1} 1_{\mathbb{A}}$$

$$= \|b\|^{1-n} \left\|M_1^{\frac{1}{2}}\right\|^2 \sum_{i=n}^{m-2} (3\|a\|^2\|b\|)^i 1_{\mathbb{A}} + \|b\|^{-n} \left\|M_1^{\frac{1}{2}}\right\|^2 (3\|a\|^2\|b\|)^{m-1} 1_{\mathbb{A}}$$

$$\longrightarrow 0_{\mathbb{A}} \ (\text{as } m,n \to \infty) \tag{7}$$

by the condition $3\|a\|^2\|b\| < 1$ and $\|b\| \geq 1$. Hence $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are Cauchy in X. Then, the completeness of $(X, \mathbb{A}, d)$, shows that there exists $u^*, v^*, w^* \in X$ that is $u_n \to u^*, v_n \to v^*$ and $w_n \to w^*$ as $n \to \infty$.

We now show that $T(u^*, v^*, w^*) = u^*, T(v^*, u^*, w^*) = v^*$ and $T(w^*, v^*, u^*) = w^*$. From Definition 2 and by condition (1), we get

$$0_{\mathbb{A}} \preccurlyeq d_b(T(u^*, v^*, w^*), u^*) \preccurlyeq b[d_b(T(u^*, v^*, w^*), u_{n+1}) + d_b(u_{n+1}, u^*)]$$

$$= b\big[d_b\big(T(u^*, v^*, w^*), T(u_n, v_n, w_n)\big) + d_b(u_{n+1}, u^*)\big]$$

$$\preccurlyeq ba^* d_b(u^*, u_n)a + ba^* d_b(v^*, v_n)a + ba^* d_b(w^*, w_n)a + bd_b(u_{n+1}, u^*)$$

$$\longrightarrow 0_{\mathbb{A}} \ (\text{as } m,n \to \infty) \tag{8}$$

So, $T(u^*, v^*, w^*) = u^*$. Similarly, $T(v^*, u^*, w^*) = v^*$ and $T(w^*, v^*, u^*) = w^*$.

Thus, (u^*, v^*, w^*) is a tripled fixed point of T.

If another tripled fixed point (p, q, r) of T exists, then

$$0_{\mathbb{A}} \preccurlyeq d_b(u^*, p) = d_b(T(u^*, v^*, w^*), \mathrm{T}(p, q, r))$$
$$\preccurlyeq a^* d_b(u^*, p)a + a^* d_b(v^*, q)a + a^* d_b(w^*, r)a,$$

$$0_{\mathbb{A}} \preccurlyeq d_b(v^*, q) = d_b(T(v^*, u^*, w^*), \mathrm{T}(q, p, r))$$
$$\preccurlyeq a^* d_b(v^*, q)a + a^* d_b(u^*, p)a + a^* d_b(w^*, r)a,$$

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

$$0_{\mathbb{A}} \preccurlyeq d_b(w^*, r) = d_b(T(w^*, v^*, u^*), \mathrm{T}(r, q, p))$$
$$\preccurlyeq a^* d_b(w^*, r)a + a^* d_b(v^*, q)a + a^* d_b(u^*, p)a,$$

which implies that

$$0_{\mathbb{A}} \preccurlyeq d_b(u^*, p) + d_b(v^*, q) + d_b(w^*, r) \preccurlyeq (\sqrt{3}a)^* (d_b(u^*, p) + d_b(v^*, q) + d_b(w^*, r))(\sqrt{3}a).$$

Thus, we have

$$0_{\mathbb{A}} \preccurlyeq \|d_b(u^*, p) + d_b(v^*, q) + d_b(w^*, r)\|$$

$$\preccurlyeq \|\sqrt{3}a\|^2 \|d_b(u^*, p) + d_b(v^*, q) + d_b(w^*, r)\|$$

$$< \frac{1}{\|b\|} \|d_b(u^*, p) + d_b(v^*, q) + d_b(w^*, r)\|$$

$$\leq \|d_b(u^*, p) + d_b(v^*, q) + d_b(w^*, r)\|$$

which is a contraction. Thus, $(p, q, r) = (u^*, v^*, w^*)$, is a unique fixed point.

Finally, we will establish that T has a unique fixed point. Since $u, v, w \in X$ are comparable, then

$$d(u, v) = d(v, w) = d(u, w) \tag{9}$$

$$0_{\mathbb{A}} \preccurlyeq d_b(v^*, w^*) = d_b(T(v^*, u^*, w^*), T(w^*, v^*, u^*))$$

$$\preccurlyeq a^* d_b(v^* w^*)a + a^* d_b(u^* v^*)a + a^* d_b(w^* u^*)a$$

$$\preccurlyeq (\sqrt{3}a)^* d_b(u^*, v)(\sqrt{3}a),$$

we have

$$\|d_b(u^*, v^*)\| \leq 3\|a\|^2 \|d_b(u^*, v^*)\|,$$

It follows from the condition $3\|a\|^2 < \frac{1}{\|b\|} \leq 1$ that $\|d_b(u^*, v^*)\| = 0$

Hence,

$$u^* = v^* \tag{10}$$

Similarly,

$$v^* = w^* \tag{11}$$

From (10) and (11), it implies that $u^* = v^* = w^*$.

Conclusion

This work obtained results on tripled fixed point in C*-algebra b-cauchy spaces. Our findings extend and improve the concept of C*-algebra-valued b-metric space from coupled fixed point to tripled fixed point theorems, which satisfies some new contractive conditions. By implication, further studies should be carried out by taking this work as a basis for improvement and applicability.

References

Bakhtin, I. A. (1989). The Contraction Mapping Principle in Quasimetric Spaces. *Functional Analysis,* **30**, 26-37.

Czerwick, S. (1993). Contraction Mappings in *b*-metric Spaces. *Acta Mathematica et Informatica Universitatis Ostraviensis*, **1**, 5-11.

Czerwick, S. (1998). Nonlinear Set-Valued Contraction Mappings in *b*-metric Spaces. Seminario Matematico e Fisico Università di Modena, **46**, 263-276.

Kirk, W. and Shahzad, N. *Fixed Point Theory in Distance Spaces*. Springer, Berlin. (2014).

Aydi, H., Felhi, A. and Sahmin S. (2015). Common Fixed Points in Rectangular *b*-metric Spaces Using (E.A) Property. *Journal of Advanced Mathematical Studies*, **8**(2), 159-169.

Kumam, P., Dung, N. V. and Hang, V. (2013). Some Equivalences Between Cone *b*-metric Spaces and *b*-metric Spaces. *Abstract and Applied Analysis*. Doi.org/10.1155/2013/573740

Ma, Z., Jiang, L. and Sun, H. (2014).

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

TREND ANALYSIS ON THE PRODUCTION OF MILLET IN KEBBI STATE, NIGERIA

***Muhammad Umar, Umar Muhammad Bala & Abubakar Ibrahim**
Department of Mathematics, Federal University Birnin Kebbi, Kebbi, Nigeria
*Corresponding author: muhammadumarjega7@gmail.com

Abstract

The purpose of this research is to perform Trend analysis on the production of millet in Kebbi State, Nigeria. The research examine the trend pattern/behavior of the data, fit linear trend, quadratic trend and exponential growth curve model and determine the model that best fit the data among the three models. The research also analyses the significance of the best fitted model and forecast the production value of milletproduction in Kebbi State. The data used for this research was secondary data which covers the period of 19 years (2000 to 2018) through Kebbi Agricultural and Rural Development Authority (KARDA) Headquarters, Kebbi State. Analysis of data was achieved through Time series plot, trend analysis, measures of accuracy and test of significance. The result obtained shows that there is an increment in the production of millet in Kebbi State for the period under study.

Keywords: Millet, Production, Trend Analysis, Forecasting

Introduction The term "millet" is used loosely to refer to several types of small seeded annual grasses, belonging to species under the five genera in the tribe *Paniceae*, namely *Panicum*, *Setaria*, *Echinochloa*, *Pennisetum*and *Paspalum*, and one genus, *Eleusine*, in the tribe *Chlorideae*(FAO, 2007). Most of the genera are widely distributed throughout the tropics and subtropics of the world. The genus *Pennisetum* for example includes about 140 species, some of which are domesticated and some are growing in the wilderness (Binuomote and Odeniyi, 2013). Millet is small-grained annual, water-weather cereal belonging to grass family. They are highly tolerant of extreme weather condition such as drought and are similarly nutritious among major cereal, such as rice and wheat. Millet is an important crop in the semiarid tropics of Asia and Africa (especially in India, Mali, Nigeria and Niger), with 97% of millet production in developing countries. The most widely grown millet is pearl millet (PennisetumGlacum). Pearl millet (Pennisetumglaucum) is the world's hardiest warm season coarse cereal crop. It can survive even on the poorest soils in the driest regions, on highly saline soils and in the hottest climate condition. India is the largest single producer of pearl millet, both in terms of area (9.3 million hectares which was 30 percent of the world area) and production (8.3 million tons, again about 37 percent). World trade in pearl millet is less than 1 percent of the production. India, USA, Argentina and China are major exporters. Virtually, all the world pearl millet production is done by the subsistence farmers and is rarely commercially traded (FAOSTAT, 2017). Millet is widely grown in the semiarid tropics of Africa and Asia and constitutes a major source of carbohydrates and proteins for people living in these areas. In addition, because of the important contribution of drought resistant crops to national food security and potential health benefits, millet is one of the most important drought-resistant crops and the 6th cereal crop in terms of world agricultural production (FAO, 2007).According to Devi *et al.* (2011), millet has resistance to pests and diseases, short growing season and productivity under drought conditions compared to major cereals. The world total production of millet grains at last count was 762,712 metric tons and the top producer was India with an annual production of 334,500 tons (43.85%) and Niger was the second producers with 108,798 metric tons and Nigeria made the third world producers with 59,994 metric tons (FAO, 2007). In many African countries, millet is often the main component of many meals and is essentially consumed as steam-cooked products ("couscous") thick porridges ("To") and thin porridges ("Ogi") that can be used as a complementary food for infants and young children, it is also used in brewing beer (Obilana, 2003). In Nigeria, kunu is a very nutritious beverage that can supply most of the nutrient requirements by the body. Kunu for millet gives the highest nourishment to the body; it has more nutritive value and is a good source of energy because of the amount of protein, normal total solids, moderate pH and acidity.Millets are highly nutritious, non-glutinous and not acid forming foods. Hence they are soothing and easy to digest. They are considered to be the least allergenic and most digestible grains available. Compared to Paddy rice, especially polished Paddy rice, millets release lesser percentage of glucose and over a longer period of time. This lowers the risk of diabetes more here. Millets are particularly high in minerals like iron, magnesium, phosphorous and potassium. Finger millet (Ragi) is the richest in calcium content, about 10 times that of Paddy rice or wheat. Millets grow well in dry regions as rain-fed crops. By eating millets, we will be encouraging farmers in dry land areas to grow crops that are best suited for those regions (Stanly and Shanmugam, 2013).The constraints of Millet production are shortage of fertilizers, lack of support from the government to boost their farming activities as well as inadequate land (Izge, 2006).There is an extensive literature on value chains of food grains mainly for cereals (rice and wheat), but very few studies examined the coarse cereals including pearl millet. Coarse cereals, like sorghum and pearl millet, assume significance in the prevailing cropping pattern in dry land areas, as they require little inputs and are more droughts tolerant as compared to other competing crops (Reddy *et al.*, 2013; HOPE Project, 2017).

Materials And Methods

The area of the study is the entire Kebbi state which is among large producers of millet in Nigeria. For the purpose of analysis, Kebbi State data on millet production, area and yield was obtained from the Kebbi Agricultural and Rural Development Authority (KARDA) Headquarters, Kebbi State and the data cover the periods of 19 years (2000 -2018).

The Statistical technique used to achieve the aim and objectives of this study is; Time series analysis which involved time series plot, trend analysis, measures of accuracy and test of significance.

In this work, the data was analyzed by fitting three (3) models namely:

i. Linear trend model (Method of least squares)
ii. Exponential growth curve model
iii. Quadratic trend model

Linear Trend Model

This method satisfies the following conditions:

1. $\sum(y_t - \hat{y}_t) = 0$ (i.e the sum of deviations of the actual values and computed/fitted values is zero).

2. $\sum(y_t - \hat{y}_t)^2$ is least (i.e the sum of the squares of deviations from the actual and computed values is least hence the name least squares). The straight line trend has an equation of the type;

$$y_t = a + bt + e_t \quad (1)$$

$$\hat{y}_t = a + bt \quad (2)$$

Where y_t = the actual value, $\hat{y}_t$ = the forecast value, t = difference in time period, and 'a' and 'b' are constants which stands for y_t intercept and slope of the line respectively.

If least squares method is applied the following normal equations are obtain:

$$\sum y_t = na + b\sum t \tag{3}$$

$$\sum ty_t = a\sum t + b\sum t^2 \tag{4}$$

The values of two constants "a" and "b" are estimated by the following two equations:

$$a = \frac{\sum y_t}{n} - b\frac{\sum t}{n} \tag{5}$$

$$b = \frac{n\sum ty_t - \sum t\sum y_t}{n\sum t^2 - (\sum t)^2} \tag{6}$$

Exponential Growth Curve Model

The equation of exponential growth curve is of the form;

$$y_t = Ae^{Bt} \tag{7}$$

Take the log of both sides to base 'e'

$$lny_t = ln(Ae^{Bt}) \tag{8}$$

$$\ln(y_t) = \ln(A) + Bt \tag{9}$$

Let $y_t = \ln(y_t)$, b = B and a = $\ln(A)$

The equation becomes;

$$y_t = a + bt \tag{10}$$

From equation (5) 'a' is obtained as;

$$a = \frac{\sum y_t}{n} - b\frac{\sum t}{n} \tag{11}$$

But, $\ln(A) = a$

From equation (6), 'b' is obtained as;

$$b = \frac{n\sum ty_t - \sum t\sum y_t}{n\sum t^2 - (\sum t)^2} \tag{13}$$

But b = B

$$B = \frac{n\sum ty_t - \sum t\sum y_t}{n\sum t^2 - (\sum t)^2} \tag{14}$$

Quadratic Trend Model

A quadratic trend changes direction once and then continues in the opposite direction throughout the rest of the series.

The equation of quadratic trend is of the form;

$$y_t = a + bt + ct^2 \tag{15}$$

If least squares method is applied the following normal equations are obtain:

$$\sum y_t = na + b\sum t + c\sum t^2 \tag{16}$$

$$\sum ty_t = a\sum t + b\sum t^2 + c\sum t^3 \tag{17}$$

$$\sum t^2 y_t = a\sum t^2 + b\sum t^3 + c\sum t^4 \tag{18}$$

Equation (16), (17) and (18) are represented below in matrix form:

$$\begin{pmatrix} n & \sum t & \sum t^2 \\ \sum t & \sum t^2 & \sum t^3 \\ \sum t^2 & \sum t^3 & \sum t^4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \sum y_t \\ \sum ty_t \\ \sum t^2 y_t \end{pmatrix} \tag{19}$$

$$\text{Let, } A = \begin{pmatrix} n & \sum t & \sum t^2 \\ \sum t & \sum t^2 & \sum t^3 \\ \sum t^2 & \sum t^3 & \sum t^4 \end{pmatrix} \tag{20}$$

$$K = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{21}$$

$$\text{A nd } B = \begin{pmatrix} \sum y_t \\ \sum t y_t \\ \sum t^2 y_t \end{pmatrix} \tag{22}$$

$$\det A = \begin{vmatrix} n & \sum t & \sum t^2 \\ \sum t & \sum t^2 & \sum t^3 \\ \sum t^2 & \sum t^3 & \sum t^4 \end{vmatrix} \tag{23}$$

$$\det A = n(\sum t^2 \sum t^4 - (\sum t^3)^2) - \sum t(\sum t \sum t^4 - \sum t^2 \sum t^3) + \sum t^2(\sum t \sum t^3 - (\sum t^2)^2) \tag{24}$$

To obtain 'a', we replace the first column of matrix A with matrix B and find its determinant.

$$\det(a) = \begin{vmatrix} \sum y & \sum t & \sum t^2 \\ \sum ty & \sum t^2 & \sum t^3 \\ \sum t^2 y & \sum t^3 & \sum t^4 \end{vmatrix} \tag{25}$$

$$\det(a) = \sum y(\sum t^2 \sum t^4 - (\sum t^3)^2) - \sum t(\sum ty \sum t^4 - \sum t^2 y \sum t^3) + \sum t^2(\sum ty \sum t^3 - \sum t^2 \sum t^2 y) \tag{26}$$

$$a = \frac{\det(a)}{detA} \tag{27}$$

To obtain 'b', we replace the second column of matrix A with matrix B and find its determinant.

$$\det(b) = \begin{vmatrix} n & \sum y & \sum t^2 \\ \sum t & \sum ty & \sum t^3 \\ \sum t^2 & \sum t^2 y & \sum t^4 \end{vmatrix} \tag{28}$$

$$\det(b) = n(\sum ty \sum t^4 - \sum t^3 \sum t^2 y) - \sum y(\sum t \sum t^4 - \sum t^2 \sum t^3) + \sum t^2(\sum t \sum t^2 y - \sum ty \sum t^2) \tag{29}$$

$$b = \frac{\det(b)}{detA} \tag{30}$$

To obtain 'c', we replace the third column of matrix A with matrix B and find its determinant.

$$\det(c) = \begin{vmatrix} n & \sum t & \sum y \\ \sum t & \sum t^2 & \sum ty \\ \sum t^2 & \sum t^3 & \sum t^2 y \end{vmatrix} \tag{31}$$

$$\det(c) = n(\sum t^2 \sum t^2 y - \sum ty \sum t^3) - \sum t(\sum t \sum t^2 y - \sum t^2 \sum ty) + \sum y(\sum t \sum t^3 - (\sum t^2)^2) \tag{32}$$

$$c = \frac{\det(c)}{detA} \tag{33}$$

RESULT AND DISCUSSION

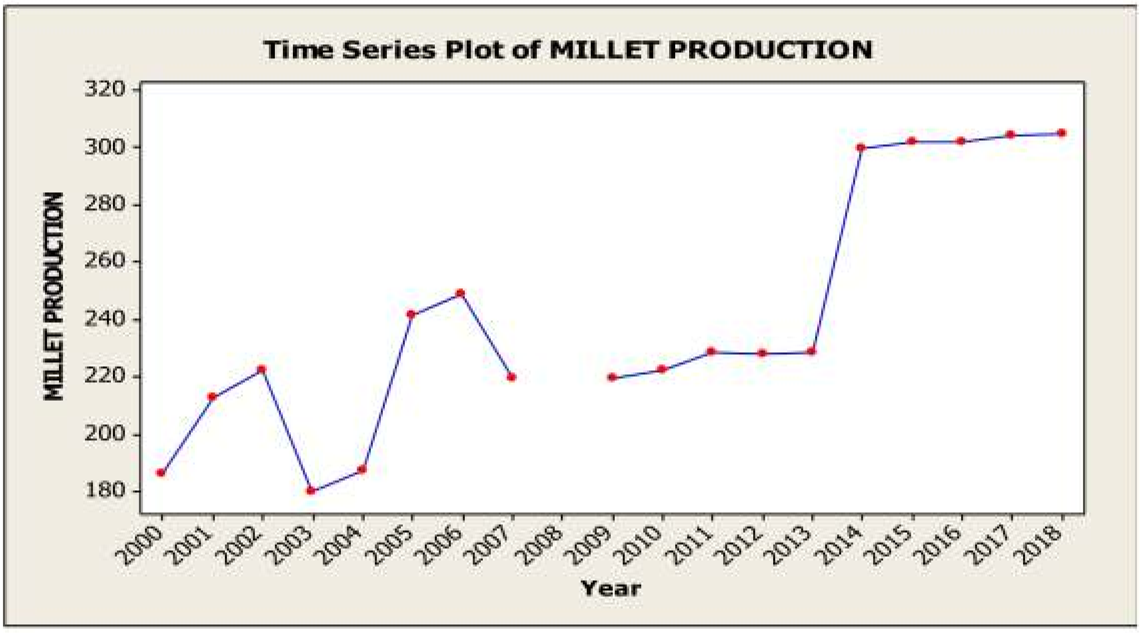


Figure 1: Time Series Plot of Millet Production with Missing Value

The figure above shows the graph for the data of millet of the production for the period under study (2000-2018). The graph also shows that the value/data for the year 2008 was missing. Therefore, there is need of estimating the missing data.

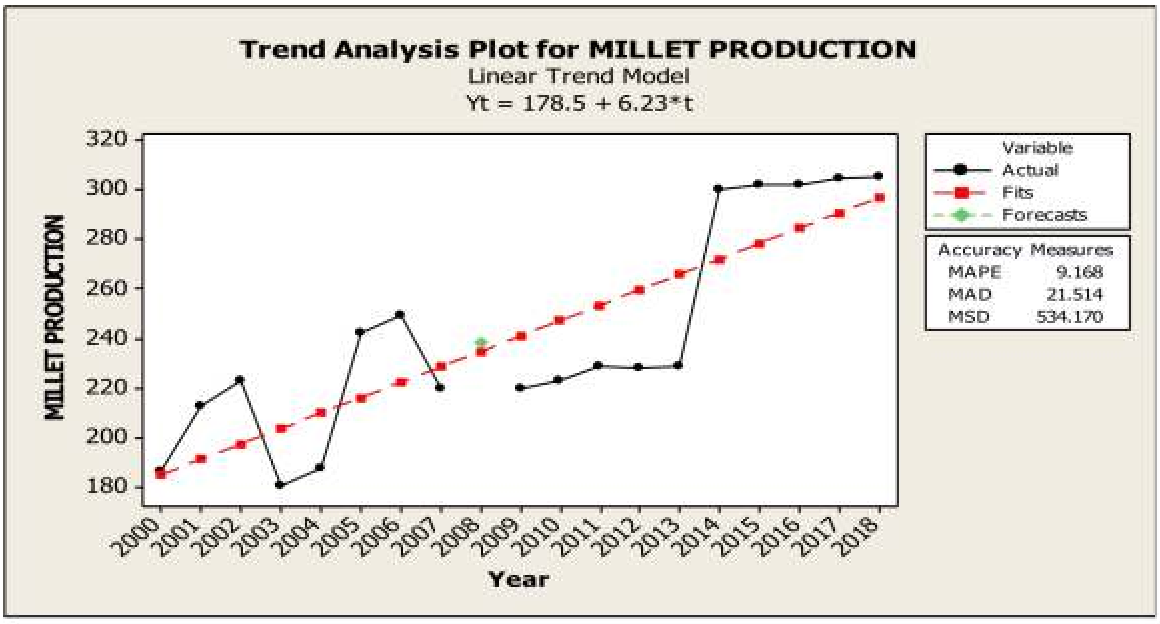


Figure 2: Linear Trend Analysis Plot for Millet Production with Missing Value

Fitted Trend Equation

Yt = 178.5 + 6.23*t

Forecasts

Period	Forecast
2008	238.088

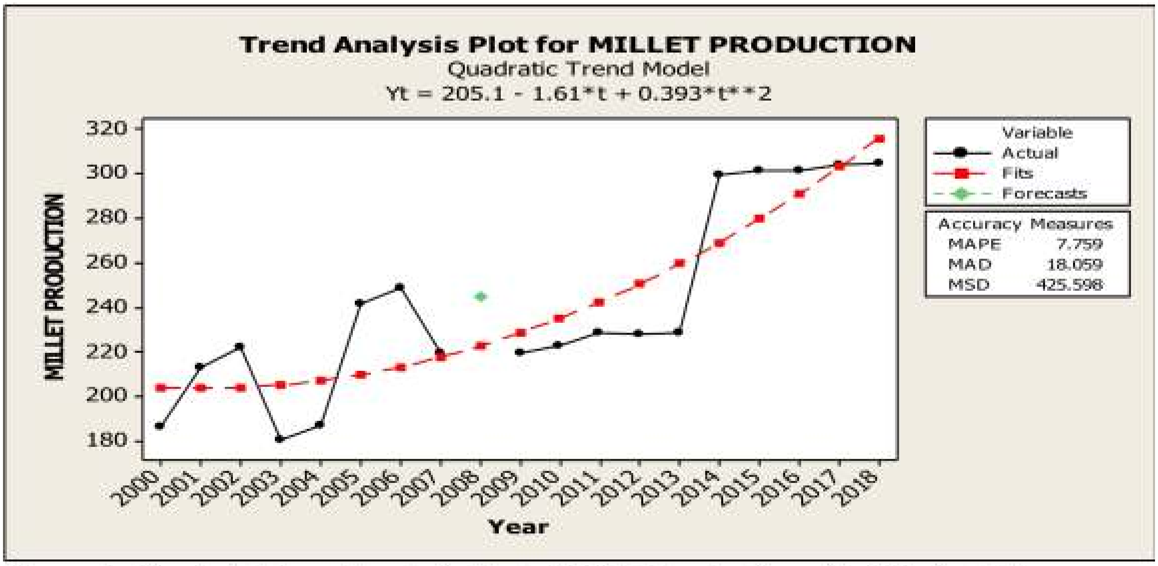


Figure 3: Quadratic Trend Analysis Plot of Millet Production with Missing Value

Fitted Trend Equation

Yt = 205.1 - 1.61*t + 0.393*t**2

Forecasts

Period	Forecast
2008	244.804

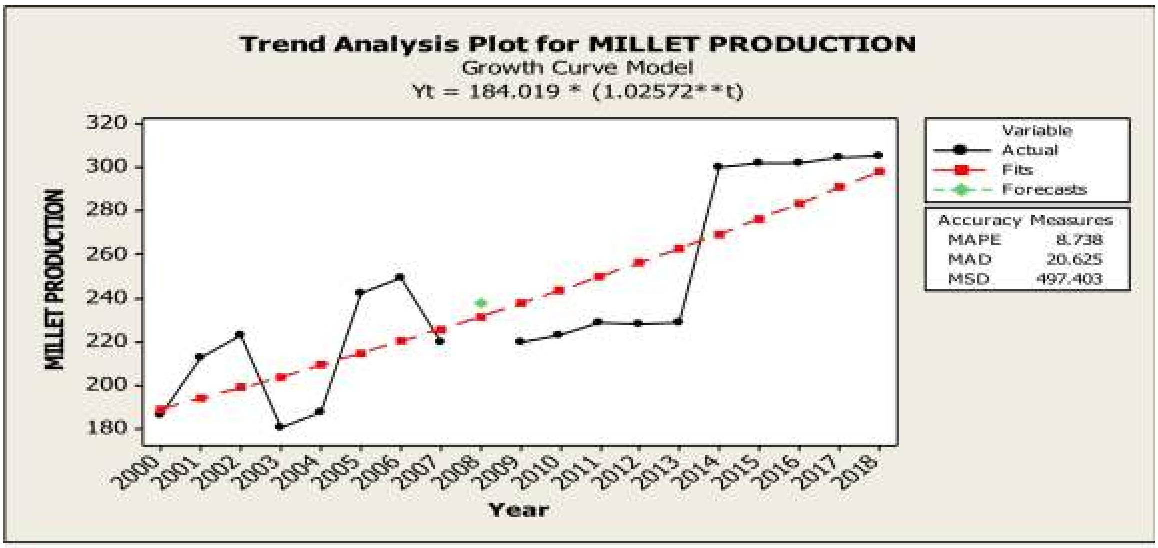


Figure 4: Exponential Growth Curve Trend Analysis Plot of Millet Production with Missing Value

Fitted Trend Equation

Yt = 184.019 * (1.02572**t)

Forecasts

Period	Forecast
2008	237.732

Table 1: Accuracy Measures for Linear, Quadratic and Exponential Growth Curve Model of Millet Production with Missing Value of the Years under Study

Accuracy	LINEAR	Quadratic	Exponential
MAPE	9.168	7.759	8.738
MAD	21.514	18.059	20.625
MSD	534.170	425.598	497.403

The table 1 above shows that after comparing the three models for millet, the quadratic trend model has the lowest accuracy measures for the crop and therefore the best model fit for the forecast (prediction) of the data for the year 2008.
Using the quadratic model which is the best fit, the forecast (predicted) value for millet production for the year 2008 is 244.804.

RESULTS AFTER ESTIMATING THE MISSING VALUE

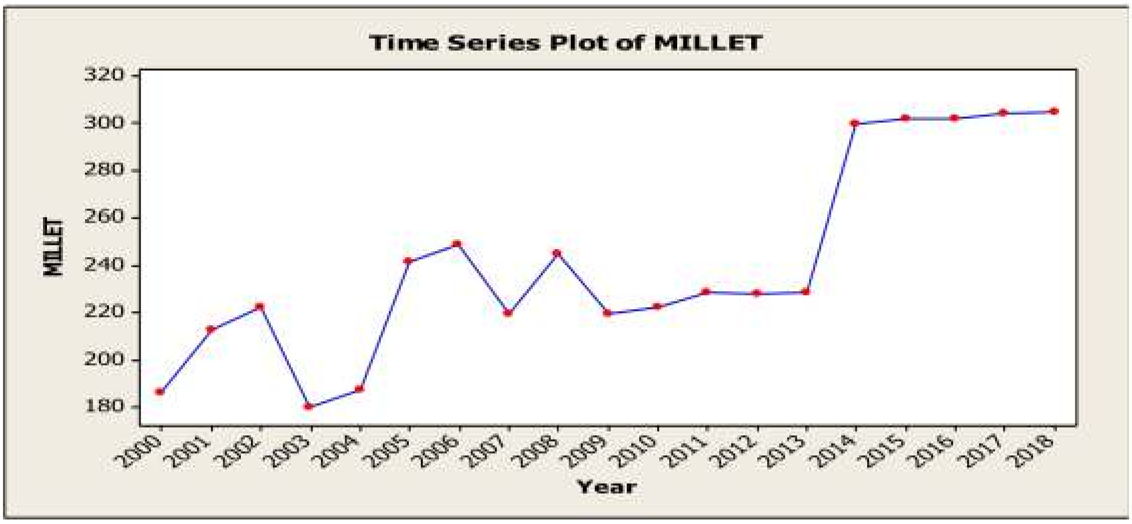


Figure 5: Time Series Plot of Millet Production
The figure 5 above shows the graph of the data of millet production for the period under study (2000- 2018). It showed an upward and downward fluctuation throughout the series. The year 2018 has the highest production figure for the period under study with a production figure of 305.2 metric tons. The lowest production figure for the period was recorded in the year 2003 with a production figure of 180 metric tons. It also shows that the high production started at year 2014.

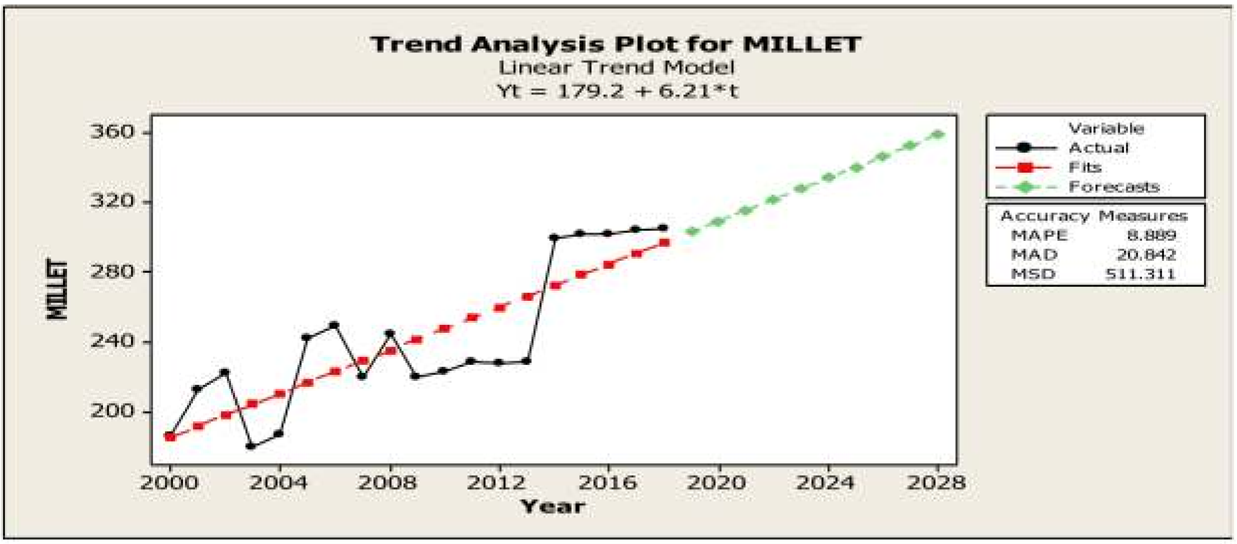


Figure 6: Linear Trend Analysis Plot for Millet Production

Fitted Trend Equation

Yt = 179.2 + 6.21*t

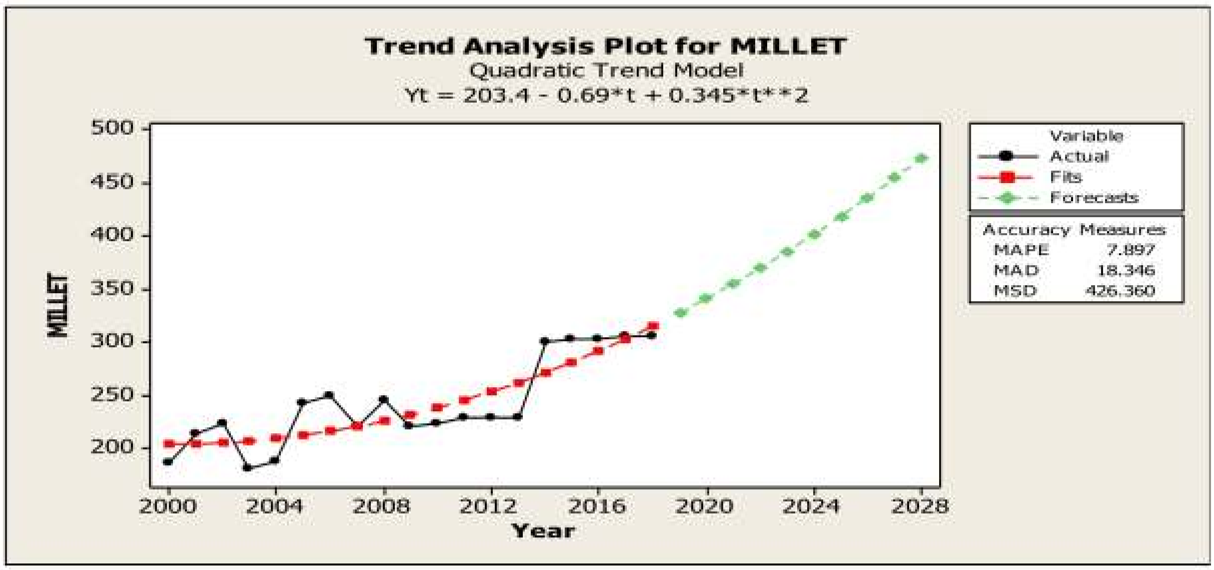


Figure 7: Quadratic Trend Analysis Plot for Millet Production

Fitted Trend Equation

$Yt = 203.4 – 0.69*t + 0.345*t^2$

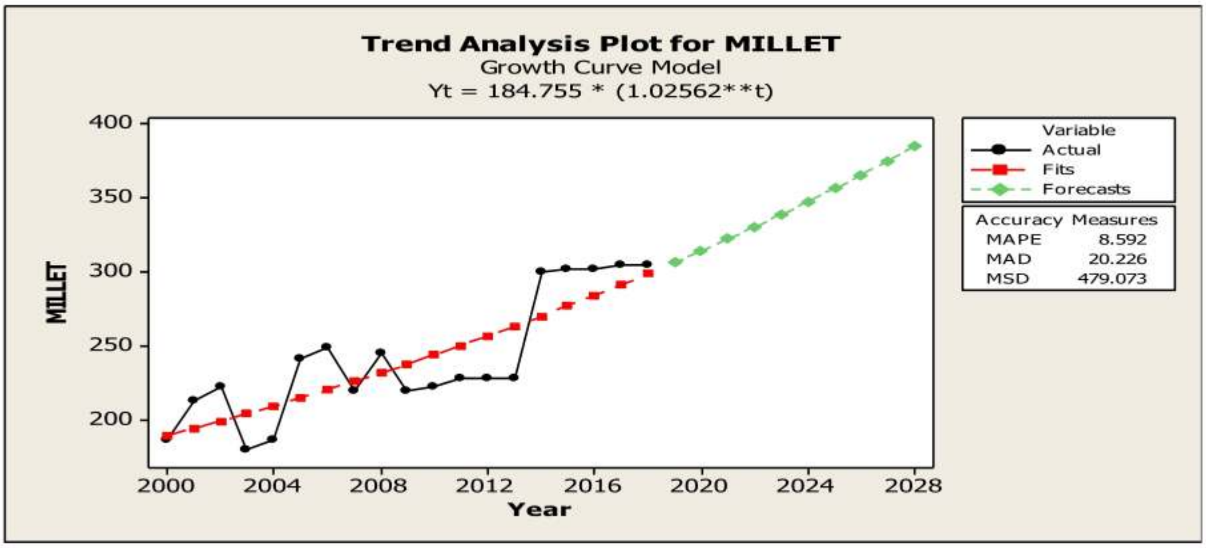


Figure 8: Exponential Growth Curve Trend Analysis Plot for Millet Production

Fitted Trend Equation

Yt = 184.755 (1.02562**t)

Table 2: Accuracy Measures for Linear, Quadratic and Exponential Trend Model for r Production

Accuracy	Linear	Quadratic	Exponential
MAPE	8.889	7.897	8.592
MAD	20.842	18.346	20.226
MSD	511.311	426.360	479.073

The table 2 above shows that after comparing the three models for millet production in Kebbi state, quadratic model has the lowest accuracy measures for the crop, therefore the is best model that fit for the forecast (prediction) of the data for the years under study.

Using the quadratic model which is the best fit, the forecast values for millet production for the year 2019, 2020, 2021, 2022, 2023, 2024, 2025, 2026, 2027 and 2028 are 327.525, 340.977, 355.119, 369.951, 385.473, 401.684, 418.586, 436.177, 454.458 and 473.429 respectively.

Test of Significance for the Best Fitted Model (Quadratic Model)

The model is; MILLET = 203.4 - 0.690 Time + 0.3449 Time**2

Where; **2 = power of 2

S = 22.5012   R-Sq = 74.4%   R-Sq(adj) = 71.2%

From the result above, the R^2 value shows that time (years) explains 74.4% of the variance in millet production, indicating that the model fits the data well.

Table 3: Analysis of Variance for Quadratic Model of the Production of Millet

Analysis of Variance					
Source	**DF**	**SS**	**MS**	**F**	**P**
Regression	2	23584.4	11792.2	23.29	0.000
Error	16	8100.8	506.3		
Total	18	31685.2			

The p-value in the table above (0.000), indicates that the relationship between time (years) and (production of millet) is statistically significant at α-level of 0.05. This implies that at least one of the coefficients of the predictor variable is not equal to zero.

Conclusion

From the analysis performed, it was observed that the trends exhibit upward and downward movement throughout the series. By fitting three (3) models; linear, exponential growth curve and quadratic model, the results of the study indicates that quadratic trend model is the best fitted model for the production of millet in Kebbi state. The results also indicate that the model fit the data well with R^2 value of 74.4% and adjusted R^2 value of 71.2% which was significant at $\alpha = 0.05$ level of significance. From the forecasted values of the best fitted model, it shows that there will be an increase in production of millet as the number of years increase in the nearest future to come. Based on the result and findings, we concluded that there is massive production of millet in Kebbi State.

References

Binuomote, S. O. & Odeniyi, K. A. (2013). Effect of Crude Oil on Agricultural Productivity in Nigeria 1981-2010. *International Journal of Applied Agricultural and Apicultural ResearchIJAAAR, 9*(1&2):131-139.

Devi, P.B., Vijayabharathi, R., Sathyabama, S., Malleshi, N.G., & Priyadarisini, V.B. (2011). Health benefits of finger millet (Eleusinecoracana L.) polyphenols and dietary fiber: a review. *Journal of Food Science Technology,* DOI: 10.1007/s13197-011-0584-9.

Food and Agricultural Organization (FAO).(2007a).*Millet Statistic*, available at http://www.milletweb.org.

Food and Agricultural Organization (FAO).(2007b). *Annual Publication* Rome, Italy: Food and Agricultural Organization.

HOPE Project. (2017).*Value chain development for pearl millet*:*Harnessing Opportunities for Productivity Enhancement (HOPE) of Sorghum and Millets in sub-Saharan Africa and South Asia*. International Crops Research Institute for Semi-Arid Tropics (ICRISAT), Hyderabad, available at: http://hope.icrisat.org/

Izge, A. U. (2006). *Combining Ability and Heterosis of Grain Yield Components among Pearl Millet (Pennisetumglaucum L. R. Br) in Breds*.PhD Thesis, Federal University of Technology, Yola, Nigeria.pp. 148.

Obilana, A. B. (2003). *Overview: Importance of Millets in Africa. World (All Cultivated Millet Species),* 38, 28.

Reddy, A.A., Yadav, O.P., Malik, D.P., Singh, I.P., Ardeshna, N.J., Kundu, K.K., Gupta, S.K., Sharma, R., Sawargaonkar, G., Moses S. D., & Reddy, S. K. (2013). Utilization pattern, demand and supply of pearl millet grain and fodder in Western India.*International Crops Research Institute for the Semi-Arid Tropics,37:* 24.

Stanly, P. J. M., & Shanmugam, A. (2013).A Study on Millets Based Cultivation and Consumption in India. *International Journal of Marketing, Financial Services and Management Research, 2(*4):1-10.

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

AN ECONOMIC ORDER QUANTITY MODEL FOR ITEMS THAT ARE BOTH AMELIORATING AND DETERIORATING WITH LINEAR INVENTORY LEVEL DEPENDENT DEMAND AND FIXED PARTIAL BACKLOGGING RATE

Y. I. Gwanda, A. A. Yusuf* and M.Z. Ringim
Department of Mathematics
Kano University of Science and Technology, Wudil, Nigeria
*Corresponding author: ayusuf07@gmail.com

Abstract

In this paper, we develop an economic order quantity model for items that are simultaneously ameliorating and deteriorating where the demand rate is a function of the on-hand inventory with shortages. The inventory undergoes two stages. In the first stage, the items incurred increase in weight or utility due to growth and at the same time may deteriorate in value due to diseases, death, feeding expenses and other factors. Thus in this stage the inventory is depleted due to the combine effects of demand and deterioration. In the second stage, shortages start to accumulate and the unsatisfied demand is partially backlogged at a rate which is a fixed fraction of demand rate during the shortage period. The model determines the best cycle length so as to minimize the overall cost. Numerical examples are given to illustrate the model and a sensitivity analysis carried out to see the effect of changes to some model parameters on the decision variables. Accordingly, the parameter values are increased/decreased taking one parameter at a time while the other parameters are kept at their original value in order to study the effect of parameter changes. It was observed that all the decision variables are sensitive to changes in all the parameters except the stock-depended demand rate and the opportunity cost per unit due to lost sales, while as expected all the decision variables increase with increase in the ordering cost.

Introduction

The decay that prevents items from being used for their original purpose is termed deterioration. Extensive literature has evolved over the years on controlling the inventory of deteriorating items. It all started with Gare and Schrader (1963) where they developed a simple economic order quantity model with a constant rate of decay. Later, researchers focused their attention on different types of models involving decaying inventories, such as Covert and Philip (1973) who extended the work of Ghare and Schrader (1963) to obtain an Economic Order Quantity (EOQ) model for a variable rate of degradation by assuming a two parameter Weibull distribution. Extensive literature on deteriorating items could be seen in the survey papers in Raafat (1991), Goyal and Giri (2001), Ruxian (2010), and Bakker and Teunter (2012).

It is also interestingly observed that some items when in inventory undergo increase in quantity or quality or both. Generally, fast growing animals like fishes, poultry, cattle, etc, provide good examples. Some fruit merchants in some tropical countries invest huge amount of money in buying large plantations of orange, banana, pineapple, etc and keep such farms for months waiting for the arrival of times of festivities when the demand for the items increases considerably high. Within this period, it is certain that these items (in the farm) undergo increase in quantity and quality. The items that exhibit such properties are mentioned as ameliorating items. The existing literature on inventory seems to ignore or give little attention to the ameliorative nature of inventory. It was not until the late 90's that Hwang (1997) for the first time studied an Economic Order Quantity (EOQ) model and a Partial Selling Quantity (PSQ) model in connection with ameliorating items under the idea that the ameliorating time follows the Weibull distribution. An EOQ model for ameliorating inventory where the lead time, the replenishment time and the demand rate are constants with no shortage of items was studied by Gwanda and Sani

(2011). The model obtained an optimum ordering quantity while keeping the relevant inventory costs minimum. This was extended by Gwanda and Sani (2012) to allow for linear trended demand.

For many stocked items, demand depends on the volume of the inventory stocked, i.e. demand tends to increase with increase in the volume of stock. It is a common knowledge that stores with larger stocks have more appeal to customers as both the quantitative and qualitative tastes are more likely to be met therein. Levin *et al.* (1992) observed that stores with large collection of goods are patronized more than the ones with smaller collection. When the store runs out of stock however, customers can place backorders and wait for resupply. Taking longer time without receiving the supply may tempt some customers to go elsewhere resulting in lost sales. These and similar observations have attracted many marketing researchers and practitioners to investigate the modeling aspects of this phenomenon. In some of the models above, the unsatisfied demand was assumed to be completely backlogged. In many cases however, demand for items is lost during the shortage period. A significant extension of classical EOQ model is the assumption that demand decreases if customers are forced to backorder. The decrease depends on the waiting time and different functional forms that have been proposed ranging from simple to complex forms to describe the scenario. Ata Allah and Nematollahi (2014) presented an inventory control problem for deteriorating items with back-ordering and financial considerations, where the paper investigates the effects of time value of money and inflation on the optimal ordering policy in an inventory control system where backordering and delay in payment are allowed.

Vandana (2018) presented an analysis of a listing model with time-dependent deterioration, ramp-type demand rate, and with complete and partial backlogging. The model presented two inventory level situations - in the first model, stock-out situation was considered as completely backlogged while in the second model, partial backlogged stock-out situation was considered. Gwanda *et al.* (2019) studied an EOQ Model for both Ameliorating and Deteriorating Items with Exponentially Increasing Demand and Linear Time Dependent Holding Cost. Vishal and Mishra (2021) developed a model of inventory with amelioration and deterioration, where the model considered price dependant rate of demand, constant rate of deterioration, varying holding cost, and total backordering.

In this paper, we develop an EOQ model for items that are simultaneously ameliorating and deteriorating with stock dependent demand and partial backlogging. The model determines the optimum cycle length so as to keep the overall costs minimum.

The proposed inventory model is developed under the following assumptions and notation:

Assumptions:

- The inventory system involves only one single item and one stocking point.
- Shortages are allowed and the excess demand is partially backlogged. The concept used in Wee (1995), where the unsatisfied demand is backlogged and the fraction of shortages backordered is $\delta, (0<\delta<1)$ is hereby used. The extreme cases $\delta = 0$ and $\delta = 1$ represent the scenarios of no shortages allowed and complete backlogging respectively.
- Amelioration and deterioration occur when the items are effectively in stock.
- The demand rate $D(t)$ at time t is assumed to be $D[(t)] = \rho + \sigma I(t)$, where ρ is a positive constant, σ is the stock dependent demand rate parameter, $0<\sigma<1$, and $I(t)$ is the non-negative inventory level at time t.

Notation:

- The cycle length is T.
- The length of time when the inventory start running into shortage is T_1.
- The inventory carrying cost in a cycle is C_h
- The unit cost of the item is a known constant C.
- The replenishment cost is also a known constant C_o per replenishment.

- Inventory holding charge per unit i, is a known constant.
- The shortage cost per unit due to backlog is C_B.
- The opportunity cost per unit due to lost sales is C_L.
- The unsatisfied demand during the cycle time (T_1, T) is I_{lost}.
- The level of non-negative inventory at any time t is I(t).
- The level of negative inventory at any time t is B(t)
- The maximum amount of demand backlogged per cycle is B,
- The total amount backordered due to backlogging in the interval (0,T) is B_T
- The initial inventory is what enters into the inventory at t = 0, and it is given by I_0.
- The ordering quantity is given by I, where $I = I_0 + B$
- The amount of non-negative inventory in the interval (0,T) is I_T.
- The rate of amelioration α is a constant.
- The rate of deterioration β is a constant
- The ameliorated amount over the cycle T when considered in terms of value (say, weight) is given by A_T.
- The total number of deteriorated units in a cycle when considered in terms of value is D_T.

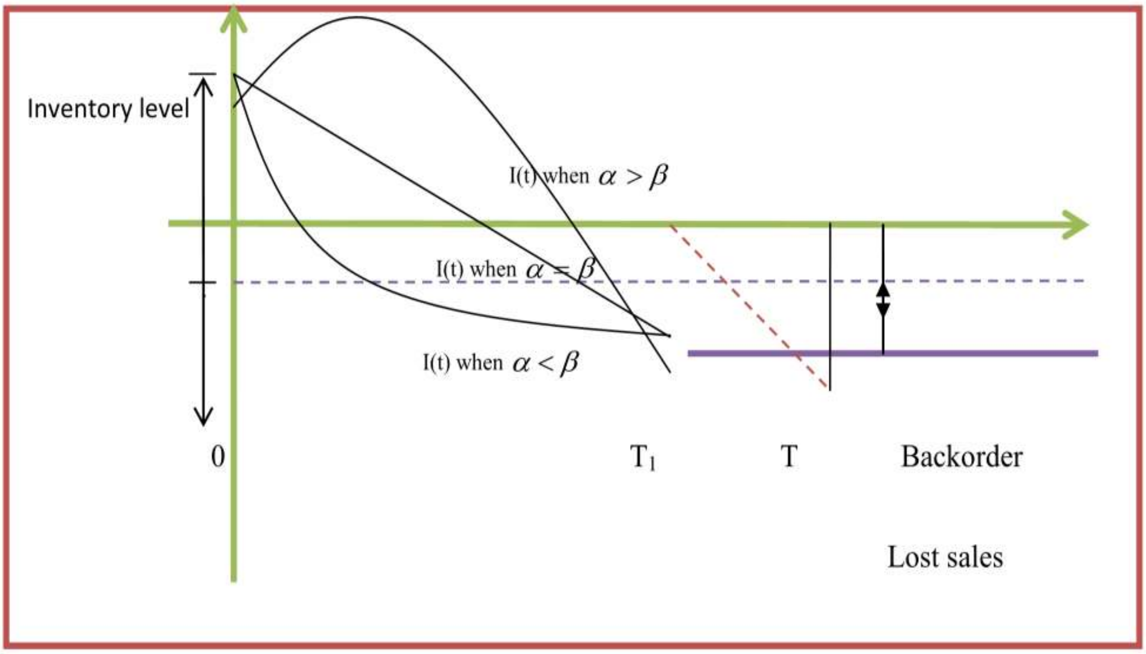


Figure 1: The graphical representation for the inventory system

Model formulation

Our objective is to determine the optimal replenishment time such that the total relevant inventory costs are kept at a minimum.

Let $I(t)$ be the on hand inventory at time $t \geq 0$, then at time $t+\Delta t$, the on hand inventory in the interval $(0,T_1)$ is given by:

$$I(t+\Delta t) = I(t) + \alpha.I(t)\Delta t - \beta.I(t)\Delta t - (\rho + \sigma I(t)).\Delta t$$

Dividing by Δt and taking limit as $\Delta t \to 0$, we obtain;

$$\frac{dI(t)}{dt} + (\beta - \alpha + \sigma)I(t) = -\rho, \; 0 \leq t \leq T_1 \tag{1}$$

The solution of Eq. (1) using the integrating factor $e^{(\beta-\alpha+\sigma)t}$ is obtained as;

$$I(t) = -\frac{\rho}{\beta - \alpha + \sigma} + k_1 e^{-(\beta-\alpha+\sigma)t}, \text{where } k_1 \text{ is a constant.} \tag{2}$$

Using $I(0) = I_0$, we obtain the value of k_1 as follows:

$$k_1 = I_0 + \frac{\rho}{\beta - \alpha + \sigma}, \tag{3}$$

The value of k_1 is then substituted in Eq. (2) to get;

$$I(t) = -\frac{\rho}{\beta - \alpha + \sigma} + \left[I_0 + \frac{\rho}{\beta - \alpha + \sigma}\right] e^{-(\beta-\alpha+\sigma)t}$$

$$= -\frac{\rho}{\beta - \alpha + \sigma} + \frac{\rho}{\beta - \alpha + \sigma} e^{-(\beta-\alpha+\sigma)t} + I_0 e^{-(\beta-\alpha+\sigma)t} \tag{4}$$

Applying the boundary condition $I(T_1) = 0$, we get:

$$0 = -\frac{\rho}{\beta - \alpha + \sigma} + \frac{\rho}{\beta - \alpha + \sigma} e^{-(\beta-\alpha+\sigma)T_1} + I_0 e^{-(\beta-\alpha+\sigma)T_1}$$

$$\Rightarrow I_0 = \frac{\rho}{\beta - \alpha + \sigma}(e^{(\beta-\alpha+\sigma)T_1} - 1) \tag{5}$$

The value of I_0 is now substituted into Eq. (4) to obtain:

$$I(t) = -\frac{\rho}{\beta - \alpha + \sigma} + \frac{\rho}{\beta - \alpha + \sigma} e^{-(\beta-\alpha+\sigma)t} + \left[\frac{\rho}{\beta - \alpha + \sigma}(e^{(\beta-\alpha+\sigma)T_1} - 1)\right] e^{-(\beta-\alpha+\sigma)t}$$

This gives; $$I(t) = \frac{\rho}{\beta - \alpha + \sigma}\left(e^{(\beta-\alpha+\sigma)(T_1-t)} - 1\right) \tag{6}$$

During the shortage interval (T_1, T), the demand at time t is partially backlogged at a fixed fraction $\delta(0 < \delta < 1)$ of the demand rate. Thus, the backorder level is governed by the differential equation below:

$$\frac{dB(t)}{dt} = -\delta(\rho + \sigma(0))$$

since there is no stock after T_1, and so

$$\frac{dB(t)}{dt} = -\delta\rho, \ T_1 \le t \le T. \tag{7}$$

The solution of Eq. (7) is obtained as follows:

$$B(t) = -\delta\rho t + k_2 \tag{8}$$

Using $B(T_1) = 0$, we have:

$$k_2 = \delta\rho T_1$$

Hence Eq. (8) becomes

$$B(t) = -\delta\rho t + \delta\rho T_1 = -\delta\rho(t - T_1) \tag{9}$$

To obtain the maximum amount of demand backlogged per cycle B, we substitute T = t in Eq. (9) and this gives

$$B = -B(T) = \delta\rho(T - T_1) \tag{10}$$

Combining Eq. (5) and Eq. (10) we obtain the total inventoried items, I as

$$I = I_0 + B$$

$$= \frac{\rho}{\beta - \alpha + \sigma}(e^{(\beta-\alpha+\sigma)T_1} - 1) + \delta\rho(T - T_1) \tag{11}$$

Total Amount of on Hand Inventory During the Complete Cycle Time T

This is given by:

$$I_T = \int_0^{T_1} I(t)dt$$

$$= \int_0^{T_1}\left[\frac{\rho}{\beta - \alpha + \sigma}\left(e^{(\beta-\alpha+\sigma)(T_1-t)} - 1\right)\right]dt$$

$$= \frac{\rho}{\beta - \alpha + \sigma}e^{(\beta-\alpha+\sigma)T_1}\int_0^{T_1} e^{-(\beta-\alpha+\sigma)t}dt - \frac{\rho}{\beta - \alpha + \sigma}\int_0^{T_1} dt$$

$$= \frac{\rho}{\beta - \alpha + \sigma}e^{(\beta-\alpha+\sigma)T_1}\left[-\frac{e^{-(\beta-\alpha+\sigma)T_1}}{\beta - \alpha + \sigma} + \frac{1}{\beta - \alpha + \sigma}\right] - \frac{\rho T_1}{\beta - \alpha + \sigma}$$

$$= -\frac{\rho}{(\beta-\alpha+\sigma)^2} + \frac{\rho}{(\beta-\alpha+\sigma)^2} e^{(\beta-\alpha+\sigma)T_1} - \frac{\rho T_1}{\beta-\alpha+\sigma}$$

$$= \frac{\rho}{(\beta-\alpha+\sigma)^2}(e^{(\beta-\alpha+\sigma)T_1} - \mathrm{T}_1(\beta-\alpha+\sigma)-1)$$

The deteriorated amounts in (0, T) is;

$$D_T = \beta I_T$$

$$= \frac{\beta\rho}{(\beta-\alpha+\sigma)^2}(e^{(\beta-\alpha+\sigma)T_1} - \mathrm{T}_1(\beta-\alpha+\sigma)-1) \qquad (12)$$

The ameliorated amount over the cycle T is given by;

$$A_T = \alpha I_T$$
$$= \frac{\alpha\rho}{(\beta-\alpha+\sigma)^2}(e^{(\beta-\alpha+\sigma)T_1} - \mathrm{T}_1(\beta-\alpha+\sigma)-1) \qquad (13)$$

The inventory holding cost in a cycle is obtained as;

$$C_h = iCI_T$$

$$= \frac{iC\rho}{(\beta-\alpha+\sigma)^2}(e^{(\beta-\alpha+\sigma)T_1} - \mathrm{T}_1(\beta-\alpha+\sigma)-1) \qquad (14)$$

Total amount backordered due to backlogging during the cycle time

This is given by

$$B_T = \int_{T_1}^{T}(-B(t))dt$$

$$= \delta\rho\int_{T_1}^{T}(t-T_1)dt$$

$$= \frac{\delta\rho}{2}(T^2-T_1^2) - \delta\rho T_1(T-T_1)$$

$$= \frac{\delta\rho}{2}\left((T^2-T_1^2) - 2T_1(T-T_1)\right) = \frac{\delta\rho}{2}(T-T_1)^2 \qquad (15)$$

Total amount of demand items unsatisfied during the cycle time

This is given by:

$$I_{lost} = \int_{T_1}^{T}\rho(1-\delta)dt = \rho(1-\delta)(T-T_1) \qquad (16)$$

Total variable cost per unit time

This is obtained as:

$TVC(T,T_1)=\frac{1}{T}$ {Ordering cost + Inventory holding cost per cycle + the deterioration cost per cycle – the amelioration cost per cycle + shortage cost per cycle due to backlog + opportunity cost per cycle due to lost sales}

$$\therefore\ TVC(T,T_1)=\frac{1}{T}\left\{C_0+iCI_T+C\beta I_T-C\alpha I_T+\frac{C_B\delta\rho}{2}(T-T_1)^2+C_L\rho(1-\delta)(T-T_1)\right\}$$

$$=\frac{1}{T}\left\{C_0+\frac{\rho(T-T_1)}{2}\left(C_B\delta(T-T_1)+2C_L(1-\delta)\right)+C(i+\beta-\alpha)I_T\right\}$$

$$=\frac{C_0}{T}+\frac{\rho(T-T_1)}{2T}\left(C_B\delta(T-T_1)+2C_L(1-\delta)\right)$$
$$+\frac{C(i+\beta-\alpha)}{T}\left[\frac{\rho}{(\beta-\alpha+\sigma)^2}(e^{(\beta-\alpha+\sigma)T_1}-\mathrm{T}_1(\beta-\alpha+\sigma)-1)\right]$$

$$=\frac{2C_0+\rho(T-T_1)\left(C_B\delta(T-T_1)+2C_L(1-\delta)\right)}{2T}$$
$$+\frac{C\rho(i+\beta-\alpha)}{(\beta-\alpha+\sigma)^2T}\left(e^{(\beta-\alpha+\sigma)T_1}-\mathrm{T}_1(\beta-\alpha+\sigma)-1\right) \tag{17}$$

Eq. (17) is a function of two variables, T_1 and T. The necessary conditions to minimize it are:

$$\frac{\partial}{\partial T_1}(TVC(T,T_1))=0 \quad\text{and}\quad \frac{\partial}{\partial T}(TVC(T,T_1))=0$$

Differentiating Eq. (17) partially with respect to T_1 we obtain:

$$\frac{\partial}{\partial T_1}(TVC(T,T_1))=\frac{\rho(C_B\delta(T_1-T)+C_L(\delta-1)}{T}+\frac{C\rho(i+\beta-\alpha)}{(\beta-\alpha+\sigma)T}(e^{(\beta-\alpha+\sigma)T_1}-1)$$

and equating to zero, we get

$$\rho\left[C_B\delta(T_1-T)+C_L(\delta-1)\right]\left(\beta-\alpha+\sigma\right)+C\rho(i+\beta-\alpha)(e^{(\beta-\alpha+\sigma)T_1}-1)=0 \tag{18}$$

Differentiating Eq. (17) partially with respect to T we obtain:

$$\frac{\partial}{\partial T}(TVC(T,T_1))=-\frac{C_0}{T^2}+\frac{\rho\left(C_B\delta(T^2-T_1^2)+2C_LT_1(1-\delta\right)}{2T^2}$$
$$+\frac{C\rho(i+\beta-\alpha)}{T^2(\beta-\alpha+\sigma)^2}(1-e^{(\beta-\alpha+\sigma)T_1}+(\beta-\alpha+\sigma)T_1)$$

and equating to zero, we get

$$-2C_0(\beta-\alpha+\sigma)^2+\rho\left(C_B\delta(T^2-T_1^2)+2C_LT_1(1-\delta)\right)(\beta-\alpha+\sigma)^2 \\ +2C\rho(i+\beta-\alpha)(1-e^{(\beta-\alpha+\sigma)T_1}+(\beta-\alpha+\sigma)T_1)=0 \tag{19}$$

To obtain the optimum values T_1^* of T_1 and T^* of T we solve the two Eq. (18) and Eq. (19) provided that,

$$\frac{\partial^2}{\partial T^2}(TVC(T^*,T_1^*))>0 \tag{20}$$

$$\det\left[H_m\left(T^*,T_1^*\right)\right]>0 \tag{21}$$

are satisfied, where $\det\left[H_m\left(T^*,T_1^*\right)\right]$ is the determinant of the Hessian matrix given by:

$$H_m\left(T,T_1\right)=\begin{bmatrix}\frac{\partial^2}{\partial T^2}(TVC(T^*,T_1^*)) & \frac{\partial^2}{\partial T\partial T_1}(TVC(T^*,T_1^*)) \\ \frac{\partial^2}{\partial T_1\partial T}(TVC(T^*,T_1^*)) & \frac{\partial^2}{\partial T_1^2}(TVC(T^*,T_1^*))\end{bmatrix} \tag{22}$$

Eq. (18) can be expressed in terms of T to obtain the following expression:

$$C_B\rho\delta T=\rho(C_B\delta T_1+C_L(\delta-1)+\frac{C\rho(i+\beta-\alpha)}{(\beta-\alpha+\sigma)}(e^{(\beta-\alpha+\sigma)T_1}-1)$$

$$T=T_1+\frac{C_L(\delta-1)}{C_B\delta}+\frac{C(i+\beta-\alpha)}{C_B\delta(\beta-\alpha+\sigma)}(e^{(\beta-\alpha+\sigma)T_1}-1) \tag{23}$$

Eq. (19) can be written as:

$$-2C_0(\beta-\alpha+\sigma)^2+C_B\rho\delta(\beta-\alpha+\sigma)^2T^2+\rho\left(2C_LT_1(1-\delta)-C_B\delta T_1^2\right)(\beta-\alpha+\sigma)^2 \\ +2C\rho(i+\beta-\alpha)\left(1-e^{(\beta-\alpha+\sigma)T_1}+(\beta-\alpha+\sigma)T_1\right)=0 \tag{24}$$

We now substitute the value of T from Eq. (23) into Eq. (24) to obtain the following:

$$-2C_0(\beta-\alpha+\sigma)^2+C_B\rho\delta(\beta-\alpha+\sigma)^2\left[\frac{(C_B\delta T_1+C_L(\delta-1))(\beta-\alpha+\sigma)+C(i+\beta-\alpha)(e^{(\beta-\alpha+\sigma)T_1}-1)}{C_B\delta(\beta-\alpha+\sigma)}\right]^2 \\ +\rho\left(2C_LT_1(1-\delta)-C_B\delta T_1^2\right)(\beta-\alpha+\sigma)^2+2C\rho(i+\beta-\alpha)\left(1-e^{(\beta-\alpha+\sigma)T_1}+(\beta-\alpha+\sigma)T_1\right)=0$$

$$\Rightarrow -2C_0(\beta-\alpha+\sigma)^2+\frac{\rho\left[(C_B\delta T_1+C_L(\delta-1))(\beta-\alpha+\sigma)+C(i+\beta-\alpha)(e^{(\beta-\alpha+\sigma)T_1}-1)\right]^2}{C_B\delta} \\ +\rho\left(2C_LT_1(1-\delta)-C_B\delta T_1^2\right)(\beta-\alpha+\sigma)^2+2C\rho(i+\beta-\alpha)\left(1-e^{(\beta-\alpha+\sigma)T_1}+(\beta-\alpha+\sigma)T_1\right)=0 \tag{25}$$

Eq. (18) is an expression in a single variable T_1 which can then be solved using any suitable numerical method. Newton-Raphson method for instance could be used to solve the equation and obtain a solution for T_1.We then substitute the value of T_1 into Eq. (23) in the solution procedure to solve for T. These solutions T^* and T_1^* jointly make the optimal solution of Eq. (17) provided Eq. (20) and Eq. (21) are satisfied. By putting the optimal values of T^* and T_1^* into Eq. (17) and Eq. (11), optimal minimum cost per unit time TVC (T_1^*, T^*) and optimal order quantity I^* are respectively obtained.

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

Numerical Examples:

Eq. (18) is used to obtain the solutions of the five numerical examples below;

Table 1: Input Parameter values for the five numerical examples

S/N	C_0	C	C_B	C_L	∝	β	σ	ρ	δ	i
1.	1200	350	150	20	0.7	0.63	0.43	500	0.86	0.42
2.	3000	350	100	10	0.77	0.53	0.35	1000	0.86	0.35
3.	2500	200	150	5	0.77	0.60	0.25	1500	0.90	0.54
4.	2000	300	90	4	0.65	0.50	0.70	1500	0.80	0.40
5.	6000	600	250	26	0.65	0.50	0.70	3000	0.80	0.40

Table 2: Output values for the five numerical examples showing the optimal solutions obtained

S/N	T_1^*	T^*	I*	$TVC(T_1^*, T^*)$	B	I_{lost}	Total shortage in the cycle
1.	54 days	99 days	129 units	9384	53	9	62
2.	108 days	160 days	424 units	13891	123	20	143
3.	63 days	96 days	384 units	19225	124	14	138
4.	49 days	98 days	370 units	15544	161	40	201
5.	49 days	77 days	606 units	62186	188	47	235

Sensitivity Analysis

Next, we carry out a sensitivity analysis to see the effect of parameter changes on the decision variables. This has been carried out on the fifth example by changing (that is, increasing or decreasing) the parameters by 1%, 5%, and 25% and taking one parameter at a time, keeping the remaining parameters at their original values. The results are as given in Table 3 below.

Table 3: Sensitivity analysis of the fifth example from Table 1 to see the effect of parameter changes

Para-meter	% change in the parameter value	% change in results					
		T_1^*	T^*	$TVC(T_1^*, T^*)$	I^*	B	I_{lost}
C_0	-25	-12	-18	-12	-1	-16	-17
	-5	0	0	-2	0	0	0
	-1	0	0	-2	-2	-3	-2
	1	0	0	0	0	0	0
	5	6	3	2	2	3	2
	25	10	11	11	12	14	13
C	-25	20	9	-9	11	-12	-13
	-5	2	0	-2	0	-4	-4
	-1	0	0	0	0	-1	-2

	1	-2	-1	0	-2	-1	2
	5	-4	-3	2	-3	1	0
	25	-14	-5	6	-7	9	9
α	-25	-31	-12	15	-15	19	23
	-5	-8	-3	4	-4	5	4
	-1	-2	0	1	-1	1	0
	1	2	1	-1	1	-1	0
	5	8	3	-4	3	-5	-9
	25	98	41	-32	50	-44	-43
β	-25	53	24	-21	28	-30	-30
	-5	8	4	3	4	-3	0
	-1	0	0	0	0	1	0
	1	-6	-3	3	-3	4	4
	5	-6	-3	3	-3	4	4
	25	-24	-9	12	-12	18	13
σ	-25	2	-4	0	0	0	0
	-5	0	0	0	0	0	0
	-1	0	0	0	0	0	0
	1	0	0	0	0	-1	0
	5	0	0	0	0	-1	0
	25	-2	1	0	-1	-2	-2
C_B	-25	-4	8	-4	-5	26	24
	-5	-2	0	-1	-1	3	2
	-1	0	0	-32	0	1	2
	1	0	-1	0	0	-1	0
	5	0	-1	1	-1	-5	-4
	25	2	-5	3	-4	-18	-17
C_L	-25	-2	1	-2	0	6	4
	-5	-2	1	0	-2	1	0
	-1	0	1	0	0	1	0
	1	0	0	0	0	0	0
	5	0	0	0	0	-2	0
	25	2	0	2	0	-5	-2
δ	-25	4	1	6	-7	-27	94
	-5	2	3	1	0	-4	21
	-1	0	0	0	0	-1	4
	1	0	0	0	0	2	-4
	5	-2	-1	-1	0	4	-21
	25	-6	-3	-5	0	25	No lost sale
‘i’	-25	37	16	-16	20	-22	-21
	-5	4	1	-3	1	-5	-4
	-1	0	0	-1	-1	-2	-2
	1	-2	-1	0	-2	-1	0
	5	-6	-3	2	-4	2	2
	25	-20	-8	10	-10	14	15

Discussion

We now discuss the effect of changes in the values of the parameters on decision variables as contained in Table 3 above. The Table shows that all the decision variables are sensitive to changes in all the parameters except σ and C_L. We notice the followings from the table:

i. T_1^* increases with increase in C_0, α, C_B, C_L, and σ but decreases with increase in C, β, ρ, δ and i
ii. T^* increases with increase in C_0, α, and σ but decreases with increase in $C, \beta, \rho, \delta, C_B$ C_L and i.
iii. I^* increases with increase in C_0, α, and ρ but decreases with increase in $C, \beta, \rho, \delta, \sigma, C_L$ and i
iv. $TVC(T_1^*, T^*)$, increases with increase in $C_0, C, C_B, C_L, \beta, \rho$ and i but decreases with increase in α and δ
v. B increases with increase in $C_0, C, \beta, \rho, \delta$ and i but decreases with increase in α, σ, C_B, and C_L
vi. I_{lost} increases with the increase in C_0, C, β, ρ, and i but decreases with increase in α, σ, C_B and C_L

The Table above shows that all the decision variables increase with increase in ordering cost. This is expected since if the ordering cost increases, the total cost $TVC(T_1^*, T^*)$ will increase and the frequency of orders will reduce so as to reduce the cost. This in effect will cause the order quantity I^* to be higher. The higher order quantity will in turn make T_1^* and T^* to be longer. This will eventually result in higher amount of backorder B and lost sales I_{lost}.

From the table also, one sees that increase in the item's cost results in decrease in the decision variables, T_1^*, T^* and I^* which is also expected since the ordering quantity I^* will have to reduce as the item's cost increases and this will relatively reduce the two periods T_1^* and T^*. Hence the backorder, B and lost sales I_{lost} will increase as it is on the table. Moreover, increase in the item's cost clearly increases $TVC(T_1^*, T^*)$.

The model has provided us with interesting scenario involving its ameliorative and deteriorative behavior where it conforms to the common expectation that amelioration and deterioration go in opposite direction. We notice from the table that as the rate of amelioration, α, increases, the ordering quantity I^* also increases resulting in higher values of the two periods T_1^* and T^*. In a similar way, backorders, B and lost sales I_{lost} will decrease due to high amount of inventory. On the other hand, as the rate of deterioration, β increases, the ordering quantity I^* decreases resulting in lower values of the two periods T_1^* and T^*. In a similar way, B and lost sales I_{lost} will increase due to low amount of inventory.

We also notice from the table that both the backlogging cost, C_B, and the lost sales cost C_L, go hand-in-hand since increase in the two quantities results in the increase in T^*, I^*, B, I_{lost} while their decrease results in the increase in $TVC(T_1^*, T^*)$ and T_1^*.

Conclusion

In this paper an economic order quantity model for both ameliorating and deteriorating items in which the demand rate is linearly dependent on inventory level with partial backlogging has been presented. The model determines the optimal quantity to order while keeping the relevant inventory costs minimum. Numerical examples are given to illustrate the developed model and sensitivity analysis carried out on the results obtained from one of the examples in order to see the effect of parameter changes on the decision variables. The sensitivity analysis shows that all the decision variables are sensitive to changes in all the parameters except σ and C_L.

References:

Ata Allah Taleizadeh and Mohammadreza Nematollahi, 2014, an inventory control problem for deteriorating items with back-ordering and financial considerations. Applied Mathematical Modelling, **38(1),** 93-109.

Bakker M. Riezebos, J, and Teunter, R.H., (2012), Review of inventory system with deterioration since 2001, *European Journal of Operation Research,* **221:** 275-284.

Covert R. F. and Philip G. C., (1973), An economic order quantity model for items with Weibull distribution, *AHE Trans.* **5:** 323-326.

Ghare P. M. and Schrader G. F., (1963), A model for exponentially decaying inventory, *Journal of Industrial Engineering,* **14:** 238-243.

Goyal S.K. and Giri B.C., (2001), Recent trends in modeling of deteriorating inventory, *European Journal of Operational Research* **134:** 1-16.

Gwanda Y. I. and Sani B., (2011), An economic order quantity model for Ameliorating Items with constant demand, *The Journal of Mathematical Association of Nigeria, ABACUS,* **38** (2):161-168.

Gwanda Y. I. and Sani B., (2012), An economic order quantity model for Ameliorating Items with Linear Trend in demand, *The Journal of Mathematical Association of Nigeria, ABACUS,* **39 (2),** 216-226.

Gwanda Y. I., Tahaa A. , Bichi A. B. and Lawan M. A., (2019), EOQ Model for both Ameliorating and Deteriorating Items with Exponentially Increasing Demand and Linear Time Dependent Holding Cost, *Global Scientific Journals*. **7(1),** 427-442.

Hwang S.K., (1997), A study of Inventory model for items with Weibull ameliorating, *Computers and Industrial Engineering* **33:** 701-704.

Levin R.I. McLaughlin, C.P., Lamone, R.P. and Kottas, J.F., (1992), Production/Operations Management: Contemporary Policy for Managing Operating Systems. *McGraw-Hill, New York.*

Raafat F., (1991), Survey of literature on continuously deteriorating inventory models, *Journal of Operational Research Society* **42**: 27-37.

Ruxian L. Hongjie L. and John R., (2010), A Review on Deteriorating Inventory Study, *Journal of Service Science & Management,* **3**: 117-129.

Vandana A., (2018), Analysis of an Inventory Model with Time-dependent Deterioration and Ramp-type Demand Rate: Complete and Partial Backlogging, *Applications and Applied Mathematics International Journal*. **13(2),** 1076-1092.

Vishal K. and Mishra P.N., (2021), Price dependent demand model for deterioration and Weibull Amelioration, *Malaya Journal of Matematik*. **9(1),** 583-586.

Wee H. M., (1995), A deterministic lot-size model for deteriorating items with shortages and a declining market. *Computers and Operations Research,* **22:** 345-356.

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

BLOCK STORMER-COWELL METHOD FOR SOLVING BRATU EQUATIONS

[1]B. T. Olabode, [2]A. L. Momoh, and [3]M. K. Duromola
([1,2,3]Federal University of Technology, Akure).

Abstract

This work focuses on the numerical solution of Bratu equations, which is extremely helpful in studying nonlinear systems. Block Stormer-Cowell-method (BSM) is proposed for the direct solution of Bratu initial and boundary value problems using boundary value techniques. The method is implemented in a block-by-block unification version which has unique advantages and is applied without restriction. The method is formulated by adopting a collocation and interpolation technique with carefully selected points within the integration interval. The stability property of the method revealed $A(\alpha)$ -stability. The rate of convergence (ROC), efficiency and solution curves are presented separately to show the proposed method's consistency, efficiency and accuracy advantages. The results show that the method gives accurate solutions and is suitable for Brat equations' direct solution.

Keywords: Bratu Equation; Stormer-Cowell-method; Block unification; $A(\alpha)$ -stability.

Introduction[DA2]

In this article, BSM is proposed for the numerical solution of Bratu initial and Bratu [DA3]boundary value problems. According to ([1], [2], [3]), this all important equation can be written as

$$u'' + \lambda e^{u} = 0 \quad 0 < x < 1$$

(1)

Subject to

$$u(0) = u(1) = 0.$$

which is considered to be boundary value problem in one dimensional coplanar coordinate. For some obvious reasons, researchers have devoted more efforts and time to the study of this type of equation. These reasons might not be unconnected with the fact that the equation appears in varieties of applications which include physical, chemical and engineering [2]. In specific term, one area of application of this equation in physical sciences can be found in thermal reaction [4]. It can also be found in chemical application such as nano-technology and fluid combustion. According to [2], engineers apply the principle of this equation in Nano-fibers and electro-spinning. Further applications of Bratu-type equation are discussed in (([5], [6]) and that of Bratu's equation in [1], [2], [3], [4],).

In literature [3] and [6] presented algorithms based on cubic spline for the solution of (1), [7] studied the approximate solution of (1) using the application of successive differentiation method, [8] examined (1) by applying variational iterative approach, while [2], [1], [4] in their separate work proposed algorithms that employed major ideal of Adomian decomposition. The work of Habtamu *et al.* [5] titled "Numerical solution of second-order initial value problems of Bratu-type equation using higher-order Rungu-Kutta method" adopted fifth-order one-step Runge-Kutta method proposed by [9] with little modification. The desire to contribute to BSM is burned out of the need for more numerical methods for the Bratu-type and Bratu equation solution without [DA4]any restriction.

The BSM considered in this article is carefully constructed to be able to tackle any equation (1) because of its nonlinearity nature. BSM is a multistep finite difference method whose development depends on constructing a continuous collocation scheme through which the main and additional methods needed to implement the BSM in multistep block unification are obtained [10]. The numerical solutions obtained in this study are presented in both 2D and 3D. We present the derivation of the proposed method in the next section of this paper with its implementation in block mode. After that, the analysis of the proposed method to establish the numerical stability, numerical example to demonstrate the efficiency advantages of the proposed method and subsequently, the conclusion drawn on the performance of the proposed method when applied to solve the numerical examples.

Mathematical Formulation of the Method[DA5]

The sole aim of this work is to derive the multistep collocation method of the form[DA6]

$$\sum_{r=3}^{k}\alpha_r(x)u_{n+r} = h^2\sum_{r=0}^{k}\beta_r(x)f_{n+r} \quad (2)$$

where $\alpha_r(x)$ and $\beta_r(x)$ are coefficients that defined the method. This shall be achieved through the interpolation and collocation of a polynomial

$$u(x) = \sum_{i=0}^{d-1}\phi_i x^i \quad (3)$$

(which are continuously differentiable) on equi-distant mesh points $\{x_j\}$. We set $r+s$ to be equal to d so as to be able to determine $\{\phi_j\}$ uniquely. We interpolate $u(x)$ and collocate $u''(x)$ at the points $\{x_{n+j}\}$ to obtain the following equations

$$u(x_{n+j}) = u_{n+j}, \quad (j = k-2, k-1) \qquad (4)$$

$$u''(x_{n+j}) = f_{n+j} \quad (j = 0(1)k) \qquad (5)$$

Note that u_{n+j} and f_{n+j} are interpolation and collocation data $u(x)$ and $u''(x)$ on $\{x_{n+j}\}$ respectively. In the light of [ ], equations (4) and (5) can be expressed in matrix-vector form as:

$$\bar{V}\phi \quad (6)$$

where $d-$ square matrix $\bar{V}$, the $p-$vectors ϕ and u are defined as follows

$$\bar{V}=\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix},\ \phi=(\phi_1,\phi_2,...,\phi_{d-1}) \text{ and } u=(u_{n+k-2},u_{n+k-1},f_n,f_{n+1},...,f_{n+k}) \quad (7)$$

Here, $\bar{V}$ are partition into $p_{11},p_{12},p_{21},p_{22}$ square matrices whose entries are generated from equation (4) and (5). We obtain a closed form of (6) by considering the inverse of the Vandermonde matrix $\bar{V}$ that is

$$\phi = Mu \quad (8)$$

where $V^{-1}=M$

We note that after the simplification of (8) and (3) equivalent continuous forms written as

$$u(x)=\alpha_{k-2}(x)u_{n+k-2}+\alpha_{k-1}(x)u_{n+k-1}+h^2\sum_{r=0}^{k}\beta_r(x)f_{n+r} \quad r=0(1)k \quad (9)$$

$$u'(x)=\frac{d}{dx}u(x) \quad (10)$$

where k is the step number, $\alpha_{k-2},\alpha_{k-1}$ and $\beta_r(x)$ are continuous coefficients are obtained. The continuous forms (9) and (10) are then used to generate the discrete and additional first derivative methods for the numerical solution of (1).

Specification of the method[DA7]

The proposed method is specified by following the procedure discussed in section two above, choosing $k=5$ and the matrix $\bar{v}$ in (6) as defined in (7) contained the following matrix partitions

$$p_{11}=\begin{pmatrix} 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 \\ 1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 \\ 0 & 0 & 2 & x_n \\ 0 & 0 & 2 & x_{n+1} \end{pmatrix},\quad p_{12}=\begin{pmatrix} x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 \\ x_{n+4}^4 & x_{n+4}^2 & x_{n+4}^6 & x_{n+4}^4 \\ 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 \\ 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 \end{pmatrix}$$

$$p_{21}=\begin{pmatrix} 0 & 0 & 0 & x_{n+2} \\ 0 & 0 & 0 & x_{n+3} \\ 0 & 0 & 2 & x_{n+4} \\ 0 & 0 & 2 & x_{n+5} \end{pmatrix},\quad p_{22}=\begin{pmatrix} 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 \\ 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 \\ 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 \\ 12x_{n+5}^2 & 20x_{n+5}^3 & 30x_{n+5}^4 & 42x_{n+5}^5 \end{pmatrix}$$

Inverting the matrix $\bar{V}$ once, using computer algebra, for example, Maple or Matlab software package, give rise to the following continuous scheme

$$u_{n+5}=\alpha_3u_{n+3}+\alpha_4u_{n+4}+h^2(\beta_0f_n+\beta_1f_{n+1}+\beta_2f_{n+2}+\beta_3f_{n+3}+\beta_4f_{n+4}+\beta_5f_{n+5}) \quad (11)$$

Where

$$\left.\begin{aligned}
\alpha_3 &= 4-N\\
\alpha_4 &= N-3\\
\beta_0 &= \frac{h^2\left(-2N^7+42N^6-357N^5+1575N^4-3836N^3+5040N^2-3176N+672\right)}{10080}\\
\beta_1 &= \frac{h^2\left(10N^7-196N^6+1491N^5-5390N^4+8400N^3-14059N+10668\right)}{10080}\\
\beta_2 &= \frac{h^2\left(-10N^7+182N^6-1239N^5+3745N^4-4200N^3-3140N+9744\right)}{5040}\\
\beta_3 &= \frac{h^2\left(10N^7-168N^6+1029N^5-2730N^4+2800N^3-5813N+13524\right)}{5040}\\
\beta_4 &= \frac{h^2\left(-10N^7+154N^6-861N^5+2135N^4-2100N^3-32N+2688\right)}{10080}\\
\beta_5 &= \frac{h^2\left(2N^7-28N^6+147N^5-350N^4+336N^3-107N-84\right)}{10080}
\end{aligned}\right\} \qquad (12)$$

The first derivative of (11) yields

$$u'_{n+5}=\frac{1}{h}\left(\alpha'_3u_{n+3}+\alpha'_4u_{n+4}+h^2\left(\beta'_0f_n+\beta'_1f_{n+1}+\beta'_2f_{n+2}+\beta'_3f_{n+3}+\beta'_4f_{n+4}+\beta'_5f_{n+5}\right)\right)$$
(13)

where

$$\left.\begin{aligned}
\alpha'_3 &= 1\\
\alpha'_4 &= 1\\
\beta'_0 &= \frac{h\left(-14N^6+252N^5-1785N^4+6300N^3-11508N^2+10080N-3176\right)}{10080}\\
\beta'_1 &= \frac{h\left(70N^6-1176N^5+7455N^4-21560N^3+25200N^2-14059\right)}{10080}\\
\beta'_2 &= \frac{h\left(-70N^6+1092N^5-6195N^4+14980N^3-12600N^2-3140\right)}{5040}\\
\beta'_3 &= \frac{h\left(70N^6-1008N^5+5145N^4-10920N^3+8400N^2-5813\right)}{5040}\\
\beta'_4 &= \frac{h\left(-70N^6+924N^5-4305N^4+8540N^3-6300N^2-32\right)}{10080}\\
\beta'_5 &= \frac{h\left(14N^6-168N^5+735N^4-1400N^3+1008N^2-107\right)}{10080}
\end{aligned}\right\} \qquad (14)$$

Equations (11) and (13) are evaluated at $N=\{0,1,2,5\}$ and $N=\{0,1,2,3,4,5\}$ respectively. Solving the resulting equations simultaneously and writing explicitly yields the following BSM

$$\left.\begin{aligned}
u_{n+1} &= u_n + u'_n + \frac{h^2}{10080}\left(2462 f_n + 4315 f_{n+1} - 3044 f_{n+2} + 1882 f_{n+3} - 682 f_{n+4} + 107 f_{n+5}\right)\\
u_{n+2} &= u_n + 2u'_n + \frac{h^2}{630}\left(355 f_n + 1088 f_{n+1} - 370 f_{n+2} + 272 f_{n+3} - 101 f_{n+4} + 16 f_{n+5}\right)\\
u_{n+3} &= u_n + 3u'_n + \frac{3h^2}{1120}\left(328 f_n + 1167 f_{n+1} - 24 f_{n+2} + 290 f_{n+3} - 96 f_{n+4} + 15 f_{n+5}\right)\\
u_{n+4} &= u_n + 4u'_n + \frac{8h^2}{315}\left(47 f_n + 2\left(89 f_{n+1} + 11 f_{n+2} + 38 f_{n+3} - 5 f_{n+4} + f_{n+5}\right)\right)\\
u_{n+5} &= u_n + 5u'_n + \frac{25h^2}{2016}\left(122 f_n + 475 f_{n+1} + 100 f_{n+2} + 250 f_{n+3} + 50 f_{n+4} + 11 f_{n+5}\right)\\
u'_{n+1} &= u'_n + \frac{h}{1440}\left(475 f_n + 1427 f_{n+1} - 798 f_{n+2} + 482 f_{n+3} - 173 f_{n+4} + 27 f_{n+5}\right)\\
u'_{n+2} &= u'_n + \frac{h}{90}\left(28 f_n + 129 f_{n+1} + 14 f_{n+2} + 14 f_{n+3} - 6 f_{n+4} + f_{n+5}\right)\\
u'_{n+3} &= u'_n + \frac{3h}{160}\left(17 f_n + 73 f_{n+1} + 38 f_{n+2} + 38 f_{n+3} - 7 f_{n+4} + f_{n+5}\right)\\
u'_{n+4} &= u'_n + \frac{2h}{45}\left(7 f_n + 32 f_{n+1} + 12 f_{n+2} + 32 f_{n+3} + 7 f_{n+4}\right)\\
u'_{n+5} &= u'_n + \frac{5h}{288}\left(19 f_n + 75 f_{n+1} + 50 f_{n+2} + 50 f_{n+3} + 75 f_{n+4} + 19 f_{n+5}\right)
\end{aligned}\right\} \quad (15)$$

Analysis of the BSM

The proposed BSM is given by the block matrix equation

$$A^{(0)}U_\mu = A^{(1)}U_{\mu-1} + h^2\left(B^{(1)}F_{\mu-1} + B^{(0)}F_\mu\right) \quad (16)$$

Where $\mu = 1,\ldots,N-1, A^{(i)}, B^{(i)}$ are 10×10 square matrix whose entries are the coefficients of (15).

$A^{(0)}$ is an identity matrix. The vectors $U_\mu, U_{\mu-1}, F_{\mu-1}$ and F_μ are defined as follows

$$U_\mu = \left(u_{n+1}, u_{n+2}, \ldots, u_{n+5}, hu'_{n+1}, hu'_{n+2}, \ldots, hu'_{n+5}\right)$$

$$F_\mu = \left(f_{n+1}, f_{n+2}, \ldots, f_{n+5}, hf'_{n+1}, hf'_{n+2}, \ldots, hf'_{n+5}\right)$$

$$U_{\mu-1} = \left(u_{n-1}, u_{n-2}, u_{n-3}, u_{n-4}, u_{n-4}, u_n, hu'_{n-1}, hu'_{n-2}, hu'_{n-3}, hu'_{n-4}, hu'_n\right)$$

$$F_{\mu-1} = \left(f_{n-1}, f_{n-2}, f_{n-3}, f_{n-4}, f_n, hf'_{n-1}, hf'_{n-2}, hf'_{n-3}, hf'_{n-4}, hf'_n\right)$$

where $y_{n-i}, y'_{n-i}, f_{n-i}, f'_{n-i}$, $i = 1(1)4$ are used to extend the zero entries of the vector notation see [10], [12] and [13]

Local Truncation error and order

With the BSM (16), we associate the linear difference operator L defined by

$$L\left[\bar{U}(x):h\right] = A^{(0)}\bar{U}_{\mu+1} + \left[A^{(1)}\bar{U}_{\mu} + h^2B^{(1)}\bar{F}_{\mu} + h^2B^{(0)}\bar{F}_{\mu+1}\right] \quad (17)$$

Where $\bar{U}_{\mu+1} = U_{\mu}, \bar{U}_{\mu} = U_{\mu-1}, \bar{F}_{\mu} = F_{\mu-1}$, and $\bar{F}_{\mu+1} = F_{\mu}$. Also $u_{n\pm j} = u(x_n \pm j), f_{n\pm j} = f(x_{n\pm j}, u_{n\pm j})$ and $y^a_{n+j} = y^a(x_n + jh)$, $a = 0(1)N$. Expanding the test function $y(x+jh)$ and its derivatives $y'(x+jh), y''(x+jh)$ as Taylor series about x, and collecting term in (17) gives the following local truncation error:

$$L\left[\bar{U}(x):h\right] = c_0\bar{U}(x) + c_1h\bar{U}'(x) + \ldots + c_ah^a\bar{U}^a(x) + \ldots \quad (18)$$

where $c_a, a = 0,1,\ldots$ are constant coefficients.

Definition: The block method (17) has algebraic order at least $p \geq 1$ provided there exists a constant $c_{p+2} \neq 0$ such that the local truncation error E_μ satisfies $\|E_\mu\| = c_{p+2}h^{p+2} + 0(h^{p+3})$, where $\|.\|$ the maximum norm [10] is.

Remarks:

a The local truncation error constants c_{p+2} of BSM (16) as defined by (18) are respectively

$$c_8 = -\begin{pmatrix} \frac{199}{24192} & \frac{19}{945} & \frac{141}{4480} & \frac{8}{189} & \frac{1375}{244192} & \frac{863}{60480} & \frac{37}{3780} & \frac{29}{2240} & \frac{8}{945} & \frac{275}{12096} \end{pmatrix}^T$$

where $c_0 = c_1 =, \ldots, c_7 = 0$

b We observed that the order of BSM (18) as obtained from the computation of the local truncation error constants are uniformly $(6,6,6,6,6,6,6,6,6,6)^T$ [11], [12], [13].

Stability of the method

The linear stability of BSM is gotten by applying (18) to test equation $y'' = \lambda^2 y$ where λ is a real constant. Let v be equal to $h\lambda$, the application of (18) to the test equation gives

$$U_\mu = \Theta(z^2)U_{\mu-1}$$

$$\Theta(z^2) = \left(A^0 + z^2B^0\right)^{-1}\left(A^1 - z^2B^1\right) \quad (19)$$

Here $\Theta\left(z^2\right)$ is called the amplification matrix and its determines the stability of the method. The stability polynomial for the BSM is gotten as

$$p(\eta,z)=\frac{\eta\left(1200\eta z^5+2962\eta z^4-20z^5+3110\eta z^3-2587z^4-18900\eta z^2+3400z^3+75600\eta z-12600z^2-151200\eta+151200\right)}{2\left(600z^5+1481z^4 1555z^3-9450z^2+37800z-75600\right)}$$

The Region of Absolute Stability (RAS) of the method is plotted using the root locus technique. The RAS is as shown the figure 1 below

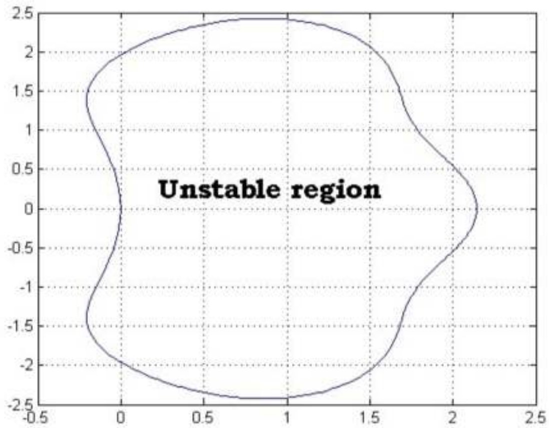


Figure 1: Region of absolute stability of the BSM

Implementation of BSM

Boundary Value Technique is adopted for implementing method (18) via a written code in Wolfram Software called Mathematical version 11.3. The block by block procedures are as itemized below

1. Choose N such that $h=\left(x_N-x_0\right)$, on the partition Q_N
2. By adopting (15), $n=0, v=5$, generate the variables $\left(y_1,y_2,y_3,y_4,y_5\right)^T$ and $\left(y_1',y_2',y_3',y_4',y_5'\right)^T$ the interval $\left[x_0,x_5\right]$and store
3. For $n=1, v=2$ generate the variables$\left(y_6,y_7,y_8,y_9,y_{10}\right)^T$ and $\left(y_6',y_7',y_8',y_9',y_{10}'\right)^T$ on the sub-interval $\left[x_6,x_{10}\right]$and store
4. Continuing the procedure for $n=2,\ldots,N-1$ and $v=3,\ldots,N$ until all the variables on the sub-intervals $\left[x_0,x_5\right],\left[x_6,x_{10}\right],\ldots,\left[x_{N-1},x_N\right]$are obtained.
5. Combine as a single block matrix equation all the block generated in steps 2 and 3 on Q_N

6 Solve simultaneously the single block matrix equation to obtain all the solution of (1) on the entire interval $[x_0, x_N]$

Numerical Experiment

In this section, our efforts shall be directed towards employing BSM as discussed above to obtain the numerical solution of some of Bratu's equations. In order to justify the efficiency and applicability of the presented method, Maximum errors are defined by

$$\text{Max Error} = \text{Max}\left\| U_{n+1} - U(x_{n+1}) \right\| \tag{20}$$

where U_{n+1} and $U(x_{n+1})$ are the numerical and exact values of U at points i in the collocation interval of points

$$\{x_1 = a, \ldots, x_i = a + (i-1)h, \ldots, x_N = b\}, \text{ for } h = \frac{|b-a|}{N-1} \tag{21}$$

The rate of convergence (ROC) is calculated using the formula

$$\text{ROC} = \log_2\left(\frac{Err2h}{Errh}\right),$$

$Errh$ is the [DA8]maximum error obtained using the step size h. In general, it is shown that the computed ROC is higher but consistent with the theoretical order 6 of the BSM.

Application of BSM to solve Bratu Equations

We first considered classical nonlinear Bratu boundary value problem in one-dimensional planar coordinates given as

$$\left.\begin{aligned} -u''(x) &= \lambda e^{u}, \ 0 < x < 1 \\ u(0) &= u(1) = 0 \end{aligned}\right\} \tag{22}$$

The Exact solution to (22) is given in [8], [5], [6], [2], [3], [7], as

$$u(x) = -2In\left[\frac{\cosh\left(\frac{x\theta}{2} - \frac{\theta}{4}\right)}{\cosh\left(\frac{\theta}{4}\right)}\right] \tag{23}$$

where θ satisfies $\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right)$.

There are three possible solutions considering the value of λ viz:

1 If $\lambda > \lambda_c$, then the Bratu problem has zero solution.

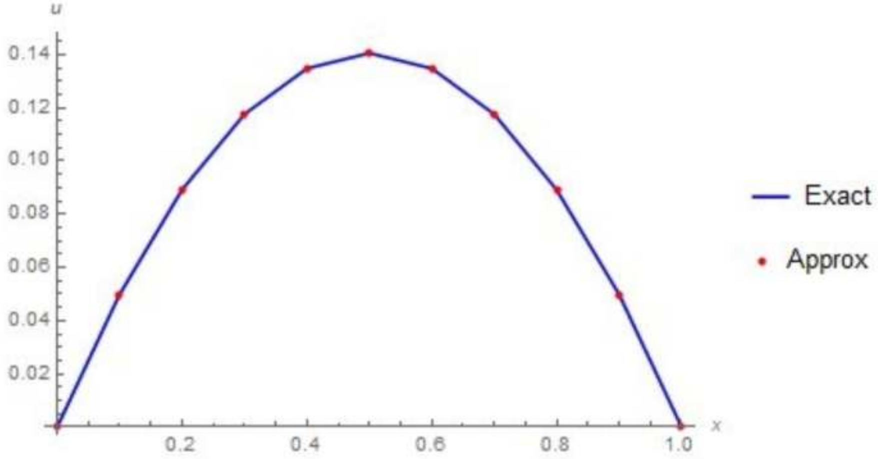
2 If $\lambda = \lambda_c$, then the Bratu problem has one solution.

3 If $\lambda < \lambda_c$, then the Bratu problem has two solutions.

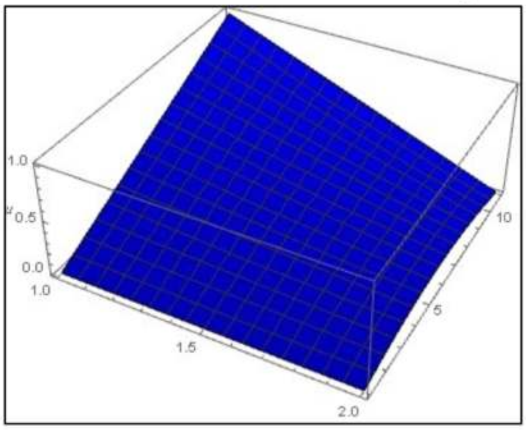
where the critical value λ satisfies the equation.

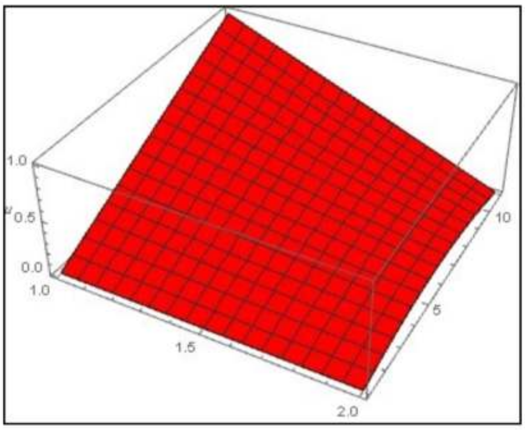
$$4 = \sqrt{2\lambda_c}\sinh\frac{\theta_c}{4}, \quad \lambda_c = 3.513830719$$

We first considered the solution of (22) for which λ is equal to 1. The solution curves are as presented in figures 2 and 3 respectively.

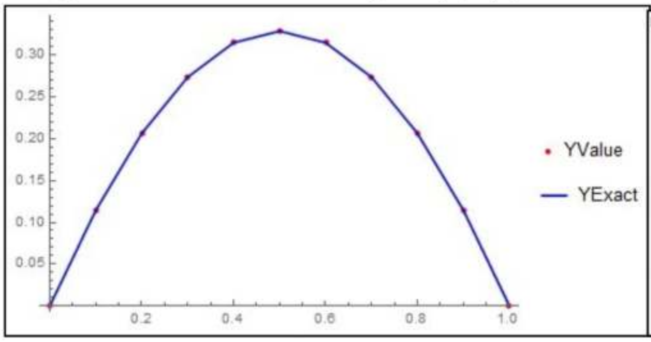


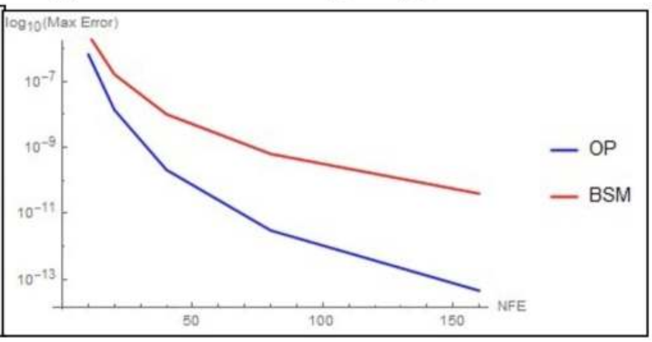
(a) 2D solution curve of Bratu eqution with λ= 1 as compared with the exact solution.





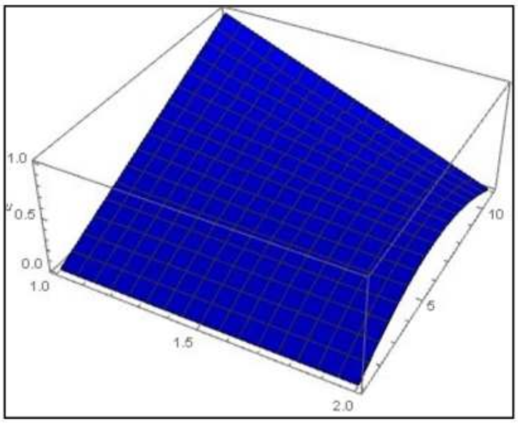
(b) 3D view of exact in figure (a) (c) 3D view of approximate value in figure (a)



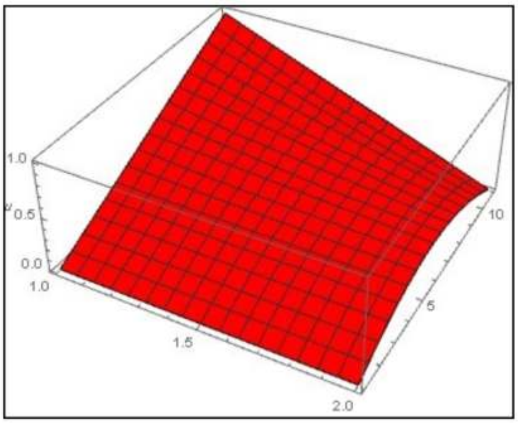


(a) 2D solution curve of Bratu eqution with λ= 2 as compared with(b) Efficiency curve of BSM with λ= 2 as compared with that of

the exact solution. OP.



(c) 3D view of exact in figure (a)



(d) 3D view of approximate value in figure (a)

Application of BSM to solve Bratu-Type Equations

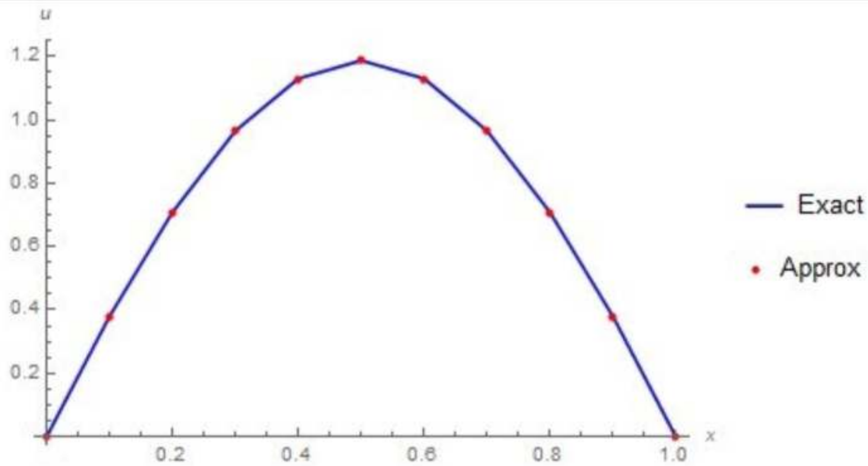
Here, Bratu-type initial value problem of the form

$$\left.\begin{aligned} u''(x) &= 2e^{u},\ \ 0 < x < 1 \\ u(0) &= u'(0) = 0. \end{aligned}\right\} \tag{24}$$

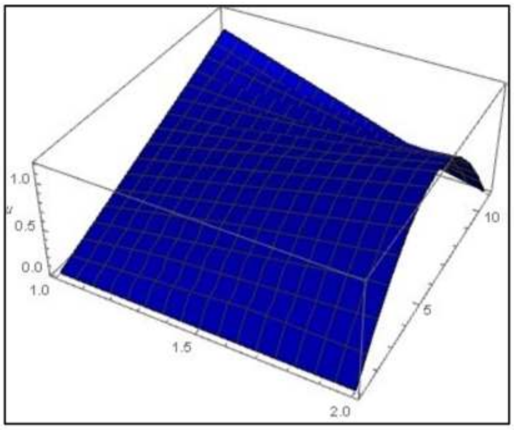
is considered to further demonstrate the efficiency of the proposed method.

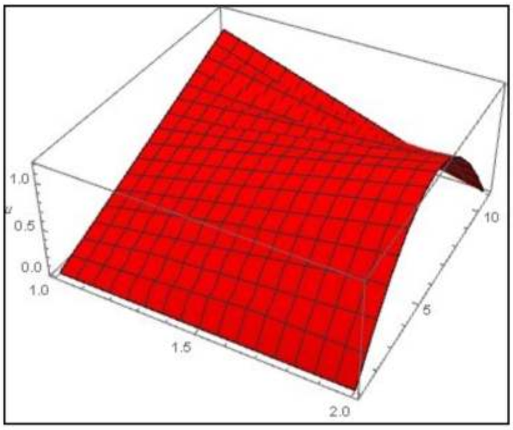
Table 1: The Maximum error and ROC of BSM with $\lambda = 1$, 2 and that of OP with $\lambda = 2$ in [6]

N	Max Error($\lambda = 1$)	ROC	Max Error($\lambda = 2$)	ROC	Maxi Error in [6]	ROC of OP in [6]
10	$2.21822\text{x}10^{-8}$		6.5722×10^{-7}		2.64(-6)	
20	4.13283×10^{-10}	5.74613	1.36604×10^{-8}	5.5883	1.64(-7)	4
40	6.68043×10^{-12}	5.95105	2.06726×10^{-10}	6.04614	1.01(-8)	4
80	1.06137×10^{-13}	5.97594	2.99544×10^{-12}	6.10881	6.31(-10)	4
160	1.69309×10^{-15}	5.97013	4.45755×10^{-14}	6.07037	3.94(-11)	4

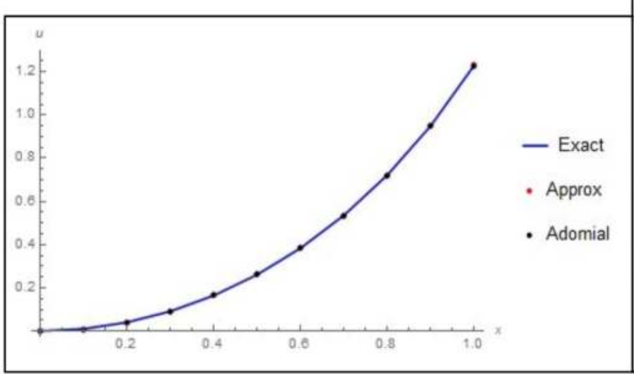


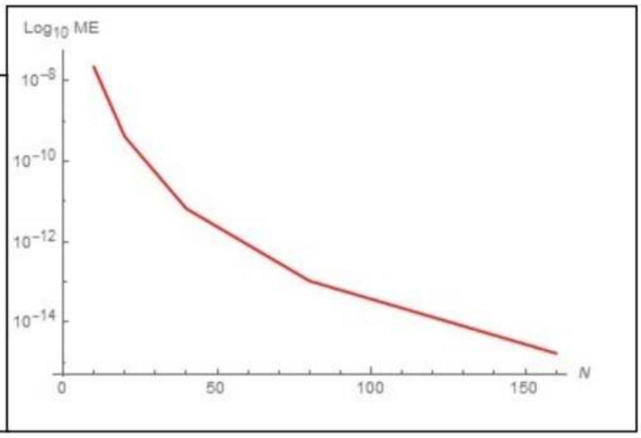
(a) 2D solution curve of Bratu eqution with $\lambda = 3.513830719$ as compared with the exact solution.





(b) 3D view of exact in figure (a) (c) 3D view of approximate value in figure (a)





(a) Numerical, Exact and Adomial solution of the Bratu-type IVP (b) Efficiency curve for Bratu equation with $\lambda = 1$

Figure 2

Conclusion

This study has investigated the numerical solution of Bratu and Bratu-type problems by constructing a block

Stomer-Cowell method. The stability study of the proposed method shows $A(\alpha)$ -stable with $\alpha = 71^{\circ}$. Numerical results of the problems under study are presented in 2D and 3D, respectively. Efficient curves for λ equal to 1 and 2 are presented to show the computational advantage of the proposed method. The convergence rate was obtained for Bratu-equation for various values of λ, and the results show that the method is consistent with the theoretical order. Our future work will present the BSM hybrid type for the numerical solution of second-order Bratu [DA9]equations.

References

Wazwaz A. M., *Adomian decomposition method for a reliable treatment of the Bratu-type equations* Applied Mathematics and Computation, vol. 166, no. 3, pp. 652-663, Jul. 2016.

Al-Mazmumy M., Al-Mutairi A., and Al-Zahrani K., *An Efficient Decomposition Method for Solving Bratu's Boundary Value Problem* American Journal of Computational Mathematics, vol. 07, no. 01, pp. 84-93, 2017.

Al-Towaiq M. and Ala'yed O., *An Efficient Algorithm based on the Cubic Spline for the Solution of Bratu-Type Equation,* Journal of Interdisciplinary Mathematics, vol. 17, no. 5-6, pp. 471-484, Nov. 2014.

Samuel O. A, Babatope E. S. and Arekete S. A. *A new result on Adomian decomposition Method for solving Bratu' problem.* Mathematical Theory and Modeling, vol.3, no.2, pp. 116-120, 2013

Habtamu G. D., Habtamu B. Y. and Solomon B. K., *Numerical solution of second order initial value problems of Bratu-type equation using higher order Rungu-Kutta method.* International Journal of Scientific and Research Publications, vol. 7, is. 10, pp. 187-197, 2017

Marwan A., Suhell K. and Ali S., *Spline-based numerical treatments of Bratu-type equation.* Palestine Journal of Mathematics, vol.1, pp. 63-70, 2012

Wazwaz A. M., *The successive differentiation method for solving Bratu equation and Bratu-type equation*

Department of Mathematics, Saint Xavier University, Chicago, IL 60655, USA. pp. 774-783, Jul. 2005.

Batiha B, *Numerical solution of Bratu-type equations by the variational iterative method.* Hacettepe Journal of Mathematics and Statistics, vol.39, no. 1, pp.23-29, 2010.

Christodoulou N. S., *An algorithm using Runge-Kutta methods of orders 4 and 5 for system of Odes.* International Journal of Numerical Methods and Application, vol.2, no. 1, pp.47-57, 2009

Jator S. N. and Oladejo H. B., *Block Nystrom Method for Singular Differential Equations of the Lane-˜ Emden Type and Problems with Highly Oscillatory Solutions* International Journal of Applied and Computational Mathematics, vol.3, s1, pp. 1385-1402, 2017

Jator S. N. and Coleman N., *A nonlinear second derivative method with a variable step-size based on continued fractions for singular initial value problems.* Applied & Interdisciplinary Mathematics vol.4, 2017, https://doi.org/10.1080/23311835.2017.1335498

Lambert J. D., *Computational Methods for Ordinary Differential Systems. The Initial Value Problems.* Wiley, Chichester, 1973.

Lambert J. D., *Numerical Methods for Ordinary Differential Systems. The Initial Value Problems.* Wiley, Chichester, 1991.

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

HYBRID INERTIAL ALGORITHM FOR GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS FOR BREGMAN RELATIVELY NONEXPANSIVE MAPPINGS IN BANACH SPACES

Lawal Umar[1], Tafida M. Kabir[2] And Ayuba M. Umar[3]
[1,2]*Department of Mathematics, Federal College of Education, Zaria, Nigeria.*
[3]*Division of Agricultural Colleges. Ahmadu Bello University, Zaria. Nigeria.*
Corresponding author: mktafida.555@gmail.com,

Abstract
In this paper, we introduce a modified hybrid inertial iterative algorithm for approximating a common solution of generalized mixed equilibrium problems and fixed points problems for finite family of continuous Bregman relatively nonexpansive mappings in Banach Spaces. Then we prove strong convergence of the sequence to some element in the mentioned set. Our results extend and improve recent results announced by many authors.

Introduction

Let E be a real Banach space with norm $\|.\|$, E^* be the dual space of E and let C be a nonempty closed convex subset of E. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunctions, where $\mathbb{R}$ is the set of real numbers, let $\psi : C \to E^*$ be nonlinear continuous monotone mapping and $\varphi : C \to \mathbb{R}$ be a convex and lower semi continuous function. The generalized mixed equilibrium problem (Darvish 2016) is to find $x \in C$ such that:

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle \psi x, y - x \rangle \geq 0, \forall y \in C. \tag{1.1}$$

The set of solutions of generalized mixed equilibrium problem (1.1) is denoted by

$$GMEP(\Theta, \psi, \varphi) = \{x \in C : \Theta(x, y) + \varphi(y) - \varphi(x) + \langle \psi x, y - x \rangle \geq 0, \forall y \in C\}.$$

The equilibrium problems are closely related with other general problems in nonlinear analysis such as fixed point problem, game theory, variational inequality problem and optimization problems. Numerous problems in optimizations, Physics and Economics can be reduced to finding solution of some equilibrium problems. Moreover, various methods have been studied for solutions of some equilibrium problems in Hilbert spaces (see (Blum and Oettli 1994, Combeltes and Hirstoaga 2005, Takahashi and Zembayashi 2009, Zhang and Cho 2016) and the references therein).

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

In 1964, an inertial algorithm was first proposed and introduced by Polyak (1964) as an acceleration process in solving a smooth convex minimisaion problem. An Inertial-types algorithm is a two-step iterative method in which the next iteration is defined by making use of the previous two iterates. Also an inertial-type algorithm plays a crucial role in speeding up the convergence of the sequence generated by the algorithm. With regards to this importance, a number of researchers have been working on an inertial-type method (see, example (Bot et.al 2015, Bot and Csetnek 2016, Chidume et.al 2018, Dong et.al 2018, Lorenz and Pock 2015) and references therein).

Bregman (1967) introduced an effective technique through Bregman distance function D_f for designing and analyzing feasibility and optimization algorithms. This opened a new area of research in which Bregman's technique is applied in various ways to iterative algorithm for solving not only feasibility problem and optimization problems, but also algorithms for solving fixed point problems for nonlinear mappings (see, example (Ali and Harbau 2016, Chang et.al 2013, Martin-Marquez et.al 2013, Ugwunnadi et.al 2014) and the reference therein).

The normalized duality mapping on E is a mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \left\{x^* \in E^* : \left\langle x, x^* \right\rangle = \| x \|^2 = \| x^* \|^2 \right\}, \forall x \in E,$$

where $\left\langle x, x^* \right\rangle$ is the pairing between element of E and that of E^*.

Let E be a reflexive Banach space, assume that $f : E \to (-\infty, +\infty]$ is a proper lower semi-continuous and convex function. We denote by dom $f := \{x \in E : f(x) < +\infty\}$, the domain of f. Let $x \in \text{int}(dom f)$; the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \left\{x^* \in E^* : f(x) + \left\langle x^*, y - x \right\rangle \le f(y), \forall y \in E\right\}.$$

The Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\left\{\left\langle x^*, x \right\rangle - f(x) : x \in E\right\}.$$

We know that the Young-Fenchel inequality holds:

$$\left\langle x^*, x \right\rangle \le f(x) + f^*(x^*), \forall x \in E, x^* \in E^*.$$

A function f on E is coercive (Hiriat and Lemarchal 1993) if the sublevel set of f is bounded; equivalently,

$$\lim_{\|x\|\to+\infty} f(x) = +\infty$$

A function f on E is said to be strongly coercive (Zelinescu 2002) if

$$\lim_{\|x\|\to+\infty} \frac{f(x)}{\| x \|} = +\infty$$

For any $x \in \operatorname{int} domf$ and $y \in E$, the right-hand derivative of f at x in the direction y is defined by

$$f^0(x,y) := \lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}$$

The function f is said to be Gateaux differentiable at x if $\lim_{t\to 0} \frac{f(x+ty)-f(x)}{t}$ exist for any y. In this case, $f^0(x,y)$ coincides with $\nabla f(x)$, the value of the gradient of f at x. The function f is said to be Gateaux differentiable if it is Gateaux differentiable for any $x \in \operatorname{int}(domf)$. The function f is said to be Frechet differentiable at x if this limit is attained uniformly in y with $\| y \| = 1$. Also f is said to be uniformly Frechet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\| y \| = 1$. It is well known that if f is Gateaux differentiable (resp. Frechet differentiable) on int (dom f), then f is continuous and its Gateaux derivative ∇f is norm-to-weak* continuous (resp. norm-to-norm continuous) on int-dom (f), (see (Asplund and Rockafellar 1969, Bonnans and Shapiro 2000)).

Definition 1.1 Bauschke et.al (2001) the function f is said to be:

(i) Essentially smooth, if ∂f is both locally bounded and single-valued on its domain;

(ii) Essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every subset of dom f;

(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 1.2. If E is a reflexive Banach space, then we have the following results:

(i) f is essentially smooth if and only if f^* is essentially strictly convex (Bauschke et.al

(2001), Theorem 5.4).

(ii) $(\partial f)^{-1} = \partial f^*$ (See Bonnans and Shapiro 2000).

(iii) f is Legendre if and only if f^* is Legendre (see Bauschke et.al (2001), Corollary 5.5).

(iv) If f is Legendre, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, ran $\nabla f = dom\nabla(f^*) = \text{int}(dom f^*)$ and ran $\nabla f^* = dom f = \text{int}(dom f)$ (see Bauschke et.al (2001), Theorem 5.10), where ran stands for the range.

Examples of Legendre function were given in (Bauschke and Borwein 1997, Bauschke et.al (2001). One important and interesting Legendre function is $\frac{1}{p}\|.\|^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space; in particular Hilbert spaces. In the rest of this paper, we always assume that $f : E \to (-\infty, +\infty]$ is Legendre.

Let $f : E \to (-\infty, +\infty]$ be a convex and Gateaux differentiable function. The function $D_f : dom \times \text{int}\, dom f \to [-\infty, +\infty)$ defined by

$$D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle$$

is called the Bregman distance with respect to f. Also the definition of D_f has the following important property (Reich and Sabah 2011):

$$D_f(z,x) := D_f(z,y) + D_f(y,x) + \langle \nabla f(y) - \nabla f(x), z - y \rangle.$$

Definition 1.3. Let $T : C \to \text{int}(dom f)$ with $F(T) \neq \phi$ be a mapping and let $F(T)$ denote the set of fixed points of T, that is., $F(T) = \{x \in C : Tx = x\}$. Then

(1) A point $p \in C$ is said to be an asymptotic fixed point of T if C contain a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Then set of asymptotic fixed points of T is denoted by $\hat{F}(T)$,

(2) quasi-Bregman nonexpansive with respect to f if,

$$F(T) \neq \phi \text{ and } D_f(p, Tx) \le D_f(p, x), \forall x \in C, p \in F(T);$$

(3) Bregman relatively non expansive with respect to f if,

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

$$\hat{F}(T) = F(T) \text{ and } D_f(p,Tx) \le D_f(p,x), \forall x \in C, p \in F(T);$$

(4) Bregman strongly nonexpansive with respect to f and $\hat{F}(T)$ if;

$$F(T) \neq \phi \text{ and } D_f(p,Tx) \le D_f(p,x), \forall x \in C, p \in \hat{F}(T);$$

and if whenever $\{x_n\} \subset C$ is bounded, $p \in \hat{F}(T)$ and

$$\lim_{n\to\infty}(D_f(p,x_n) - D_f(p,Tx_n)) = 0,$$

it follows that

$$\lim_{n\to\infty} D_f(x_n, Tx_n) = 0$$

(5) Bregman firmly nonexpansive (BFNE) with respect to f if, for all $x, y \in C$,

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \le \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle,$$

equivalently,

$$\begin{aligned} &D_f(Tx,Ty) + D_f(Ty,Tx) + D_f(Tx,x) + D_f(Ty,y) \le D_f(Tx,y) \\ &+ D_f(Ty,x). \end{aligned} \tag{1.2}$$

Definition 1.4. A mapping $T : C \to C$ is said to be closed, if for any sequence $\{x_n\} \subset C$ with $x_n \to x \in C$ and $Tx_n \to y (y \in C)$, then $y = Tx$.

Several results for fixed point approximations of Bregman nonexpansive mappings and their generalizations are established see for example (Ali and Harbau 2016, Chang et.al 2013, Darvish et.al 2019, Kazmi et.al 2018, Ali et.al 2014).

Agha *et.al* (2017) introduced an iterative process which converges strongly to a common element of the sets of solutions of finite family of generalized equilibrium problems, sets of fixed points of finite family of continuous relatively nonexpansive mappings and the sets of finite family of γ-inverse strongly monotone mappings in Banach space as follows. Let the sequences $\{x_n\}$, be generated by the following algorithm:

$$\begin{cases} x_0 \in C_0 = C, chosenarbitrary \\ z_n = \prod_C J^{-1}(Jx_n - \lambda_n A_{n+1} x_n); \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JTx_{n+1} z_n), \\ u_n \in C \ni f_1(u_n, y) + \langle B_1 y_n, y - u_n \rangle \\ + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C \\ v_n \in C \ni f_2(\upsilon_n, y) + \langle B_2 y_n, y - \upsilon_n \rangle \\ + \frac{1}{r_n} \langle y - \upsilon_n, J\upsilon_n - Jy_n \rangle \geq 0, \forall y \in C, \\ w_n = J^{-1}(\beta Ju_n + (1-\beta) J\upsilon_n), \\ C_{n+1} = \{ z \in C_n : \phi(z, w_n) \leq \phi(z, x_n) \}, \\ x_{n+1} = \prod c_{n+1}(x_0), \forall_n \geq 0, \end{cases}$$

Then, the sequence $\{x_n\}$ converges to some element of F.

Recently, Alansari et.al (2020) studied the following inertial iterative algorithm for variational inequality problem, generalized equilibrium problem and fixed point problem in Banach space. Let the sequence $\{x_n\}$ and $\{z_n\}$ be generated by the algorithm:

$$\begin{cases} x_0 = x_1, z_0 \in C, C_0 := C; \\ w_n = x_n + \theta_n (x_n - x_{n-1}) \\ y_n = \prod_C J^{-1}(Jw_n - \mu_n Dw_n); \\ u_n = J^{-1}(\alpha_n Jz_n + (1-\alpha_n) JTy_n); \\ z_{n+1} = Tr_n u_n, \\ C_n = \{ z \in C : \phi\langle z, z_{n+1} \rangle \leq \alpha_n \phi(z, z_n) + (1-\alpha_n)\phi(z, w_n) \}; \\ Q_n = \{ z \in C : \langle x_n - z, Jx_n - Jx_0 \rangle \leq 0 \}; \\ x_{n+1} = \prod C_n \cap Q_n, \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\} \subset [0,1], r_n \subset [a, \infty)$ for some $a > 0, \{\theta_n\} \subset (0,1)$ and $\{\mu_n\} \subset (0, \infty)$.

Then sequences $\{x_n\}$, converges strongly to a point $\hat{x} \in \Gamma$.

Very recently, Jantakan and Kaewcharoen (2021) proposed a new iterative method for solving the mixed equilibrium problems and fixed point problems for a countable family of Bregman relatively nonexpansive mappings in reflexive Banach space.

$$\begin{cases} x_1 \in C, T_i x_1 = z_1^i \in C; \\ u_n^i = \nabla f^*(\alpha_n \nabla f(z_n^i) + (1-\alpha_n)\nabla f(T_i x_n)); \\ z_{n+1}^i = \mathrm{Res}_{G,\varphi}^f(u_n^i); \\ C_n^i = \{z \in C : \phi(z, z_{n+1}^i) \le \alpha_n \phi(z, z_1^i) + (1-\alpha_n)\phi(z, x_n)\}, \\ C_n = \cap_{i=1}^N C_n^i; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - f(x_n), z - x_n \rangle \le 0\}, \\ x_{n+1} = proj_{C_n \cap Q_n}^f x_1, \forall n \ge 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in [0,1] such that $\lim_{n\to\infty} \alpha_n = 0$. Then, the sequence $\{x_n\}$ converges strongly to $proj_\Omega^f x_1$, where $proj_\Omega^f x_1$ is the projection of C onto Ω.

In this paper, motivated and inspired by the results of Agha et.al (2017), Alansari et.al (2020) and Jantakan and Kaewcharoen (2021) mentioned above, we study a modified inertial algorithm for approximating a common solution of generalized mixed equilibrium problems and fixed points problem for Bregman relatively nonexpansive mappings in Banach Spaces. Our results extend and improve recent results announced by many authors.

Preliminaries

In this section, we shall consider some Bregman projection and results which will be used in the proof of our main result.

Let E be a reflexive Banach space and $f : E \to (-\infty, +\infty]$ be a Gateaux differentiable and convex function. The Bregman projection of $x \in \operatorname{int} domf$ onto a nonempty, closed and convex set $C \subset domf$ is the necessarily unique vector $proj_C^f(x)$ satisfying

$$D_f(proj_C^f(x), x) := \inf\{D_f(y, x) : y \in C\}.$$

The modulus of total convexity of f at $x \in \operatorname{int} domf$ is the function $\upsilon_f(x,t) : [0,+\infty) \to [0,+\infty)$ defined by

$$\upsilon_f(x,t) := \inf\{Df(y,x) : y \in domf, \|y - x\| = t\}.$$

The function f is called totally convex at x, if $\upsilon_f(x,t) > 0$ whenever $t > 0$. The function f is called totally convex, if it is totally convex at any point $x \in \operatorname{int} domf$ and it is said to be totally convex on bounded sets, if $\upsilon_f(B,t) > 0$, for any nonempty bounded subset B of E and $t > 0$,

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

where the modulus of the total convexity of the function f on the set B is the function $\upsilon_f : \text{int}\, domf \times [0,+\infty) \to [0,+\infty)$ defined by

$$\upsilon_f(B,t) := \inf\{\upsilon_f(x,t) : x \in B \cap domf\}.$$

We know that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets by Butnariu and Resmerita (2006).

Lemma 2.1 Butnariu and Resmerita (2006). Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a Gateaux differentiable and totally convex function and let $x \in E$. Then:

(i) $z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0, \forall y \in C$;

(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y,x), \forall y \in C$.

Let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function. Following Alber (1996) and Censor and Lent (1981), let the function $V_f : E \times E^* \to [0,+\infty)$ associated with f is defined by

$$V_f(x,x^*) = f(x) f^*(x^*) - \langle x, x^* \rangle, \forall x \in E, x^* \in E^*.$$

Then V_f is nonnegative and the following assertions hold:

(1) $V_f(x,x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in E$ and $y^* \in E^*$

(2) $V_f(x,x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V_f(x, x^* + y^*)$ for all $x \in E$ and $y^* \in E^*$.

Lemma 2.2. Naraghired and Yao (2013). Let E be a Banach space $f : E \to \mathbb{R}$ be a Gateaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n \in N}$ and $\{y_n\}_{n \in N}$ be bounded sequences in E. Then

$$\lim_{n\to\infty} D_f(x_n, y_n) = 0 \text{ if and only if } \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Lemma 2.3. Butnariu and Resmerita (2006). If $x \in domf$, then the following statements are equivalent:

(i) The function f is totally convex at x,

(ii) for any sequence $\{y_n\} \subset domf$,

$$\lim_{n\to+\infty} D_f(y_n, x) = 0 \Rightarrow \lim_{n\to+\infty} \|y_n - x\| = 0.$$

Recall that the function f is called sequentially consistent Butnariu and Resmerita (2006), if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded

$$\lim_{n\to+\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to+\infty} \|y_n - x_n\| = 0.$$

Lemma 2.4. Butnariu and Lusem (2000). Let $f: E \to \mathbb{R}$ be a convex function whose domain contains at least two points. Then f is sequentially consistent if and only if it is totally convex on bounded sets.

Lemma 2.5. Reich and Sabah (2010). Let $f: E \to \mathbb{R}$ be a Gateaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.

Lemma 2.6. Kazmi et.al (2018). Let $f: E \to \mathbb{R}$ be a Legendre function and C be a nonempty, closed and convex subset of $\operatorname{int} domf$. Let $T: C \to C$ be Bregman quasi nonexpansive mapping with respect to f. Then $F(T)$ is closed and convex.

Lemma 2.7. Reich and Sabah (2010). Let $f: E \to \mathbb{R}$ be a Gateaux differentiable and totally convex function, $x_0 \in E$ and C be a nonempty, closed and convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belong to C. If $D_f(x_n, x_0) \le D_f(P_C^f x_0, x_0)$ for any $n \in N$, then $\{x_n\}$ strongly converges to $P_C^f x_0$.

The following two results are well known; (see (Zalinescu 2002))

Theorem 2.8. Let E be a reflexive Banach space and let $f: E \to \mathbb{R}$ be a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent:

(1). f is strongly coercive and uniformly convex on bounded subsets of E;

(2). $domf^* = E^*, f^*$ is bounded on bounded subsets and uniformly smooth on bounded subsets of E^*

(3). $domf^* = E^*, f^*$ is Frechet differentiable and ∇f^* is norm-to-norm uniformly continuous on bounded subsets of E^*.

Theorem 2.9. Let E be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent:

(1). f is bounded on bounded subsets and uniformly smooth on bounded subsets of E;

(2). f^* is Frechet differentiable and f^* is uniformly norm-to-norm continuous on bounded subsets of E^*.

(3). $domf^* = E^*, f^*$ is strongly coercive and uniformly convex on bounded subsets of E^*.

Lemma 2.10. Reich and Sabah (2009). Let $f : E \to \mathbb{R}$ be a uniformly Frechet differentiable and bounded on bounded subsets of E. Then, f is uniformly continuous on bounded subsets of E and ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.

In order to solve generalized equilibrium problems, we shall consider the following assumptions Blum and Oettli (1994):

The bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the following assumptions:

(A_1). $\Theta(x,x) = 0, \forall x \in C$;

(A_2). Θ is monotone, that is $\Theta(x,y) + \Theta(y,x) \le 0, \forall x,y \in C$;

(A_3). For each $x,y,z \in C, \limsup_{t \to 0} \Theta(tz + (1-t)x, y) \le \Theta(x,y)$;

(A_4). For each $x \in C, y \to \Theta(x,y)$ is convex and lower semi continuous.

Lemma 2.11. Ali et.al (2019). Let C be a nonempty, closed convex subset of a real reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a convex, continuous and strongly coercive function which bounded on bounded subsets and uniformly convex on bounded subsets of E.

Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions (A_1) – (A_4), let $\psi : C \to E^*$ be a monotone mapping and $\varphi : C \to R$ be a convex and lower semi-continuous function. For $r > 0$, a mapping $T_r : E \to C$ is defined by

$$T_r(x) = \left\{ z \in C : \tau(z,y) + \frac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x) \rangle \ge 0, \forall y \in C \right\}, \forall x \in E$$

where

$$\tau(z,y)=\Theta(z,y)+\langle \psi z, y-z\rangle+\varphi(y)-\varphi(z).$$

Then the following holds:

(a). T_r is single-valued;

(b). T_r is Bregman firmly nonexpansive type mapping (BFNE);

(c). $F(T_r)=GMEP(\Theta,\psi,\varphi)=\hat{F}(T_r)$;

(d). $GMEP(\Theta,\psi,\varphi)$ is closed and convex;

(e). $D_f(p,T_r(w))+D_f(T_r(w),w)\leq D_f(p,w),\forall p\in F(T_r), w\in E$.

Main Results

Theorem 3.1. Let C be a nonempty closed and convex subset of a reflexive Banach space E and $f:E\to\mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of E. Let $\Theta_1,\Theta_2:C\times C\to\mathbb{R}$, $k=1,2$, be bifunctions which satisfying assumptions (A_1) – (A_4), $\psi_1,\psi_2:C\to E^*$, $k=1,2.$ be continuous monotone mappings and $\varphi_1,\varphi_2:C\to\mathbb{R}$, $k=1,2.$ be convex and lower semi-continuous functions. Let $T_j:E\to E, j=1,2,\ldots,d$ be finite family of continuous Bregman relatively nonexpansive mappings, assume that $\Omega:=(\cap_{k=1}^{2}GMEP(\Theta_k,\psi_k,\varphi_k))\cap(\cap_{j=1}^{d}F(T_j))\neq\phi$. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by the iterative schemes:

$$\begin{cases} x_0, z_0 \in C_0 = E \\ w_n = \nabla f^*\left(\nabla f(x_n) + \theta_n(\nabla f(x_n) - \nabla f(x_{n-1}))\right); \\ y_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(T_j w_n)); \\ u_n \in C \ni \Theta_1(u_n, y) + \langle \psi_1 y_n, y - u_n \rangle + \varphi_1(y) - \varphi_1(u_n) \\ + \dfrac{1}{r_n}\langle y - u_n, \nabla f(u_n) - \nabla f(y_n)\rangle; \\ v_n \in C \ni \Theta_2(\upsilon_n, y) + \langle \psi_2 y_n, y - \upsilon_n \rangle + \varphi_2(y) - \varphi_2(\upsilon_n) \\ + \dfrac{1}{r_n}\langle y - \upsilon_n, \nabla f(\upsilon_n) - \nabla f(y_n)\rangle; \\ z_{n+1} = \nabla f^*(\beta_n \nabla f(u_n) + (1-\beta_n)\nabla f(\upsilon_n)); \\ C_{n+1} = \left\{z \in C_n : D_f(z, z_{n+1}) \le \alpha_n D_f(z, z_n) + (1-\alpha_n) D_f(z, w_n)\right\}; \\ x_{n+1} = proj^f_{C_{n+1}} x_0, \forall n \ge 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\},\{\beta_n\}$ are sequences in $[0,1]$, such that $\lim\limits_{n\to\infty}\alpha_n = 0, \{r_n\} \subset [a,\infty)$, for some $a > 0$ and $\theta_n(x_n - x_{n-1})$ is the inertial term with $\theta_n \in (0,1)$. We shall define

$$T_{k,r}(x) = \left\{ z \in C : \Theta_k(z,y) + \langle \psi_k x, y - z\rangle + \varphi_k(y) - \varphi_k(z) + \frac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x)\rangle \ge 0, \forall y \in C \right\},$$

$\forall x \in E, k = 1,2.$

Then, $\{x_n\}$ converges strongly to $proj^f_{\Omega} x_0$, where $proj^f_{\Omega} x_0$ is the Bregman projection of C onto Ω.

Proof. We divide the proof into a number of steps:

Step 1: We show that $\Omega := (\cap_{k=1}^2 GMEP(\Theta_k, \psi_k, \varphi_k)) \cap (\cap_{j=1}^d F(T_j))$ is closed and convex. It follows from Lemma 2.6 that $\cap_{j=1}^d F(T_j)$ is closed and convex. Also from Lemma 2.11(d) that $(\cap_{k=1}^2 GMEP(\Theta_k, \psi_k, \varphi_k)$ is closed and convex. Therefore $\Omega := (\cap_{k=1}^2 GMEP(\Theta_k, \psi_k, \varphi_k)) \cap (\cap_{j=1}^d F(T_j))$ is closed convex.

Step 2: We show that C_{n+1} is closed and convex for all $n \ge 0$. Now, by the assumption that $C_0 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \ge 0$. Let $a, b \in C_{n+1}$ and $z = ta + (1-t)b \in C_{n+1}$, where $t \in [0,1]$ then, we have

$$D_f(a,z_{n+1}) \le \alpha_n D_f(a,z_n) + (1-\alpha_n) D_f(a,w_n)$$

and

$$D_f(b,z_{n+1}) \le \alpha_n D_f(b,z_n) + (1-\alpha_n) D_f(b,w_n).$$

Recall that $D_f(y,x) = f(y) - f(x) - \langle \nabla f(x), y-x \rangle$. Now, using this definition the above two inequalities are equivalent to

$$\alpha_n \langle \nabla f(z_n), a-z_n \rangle + (1-\alpha_n) \langle \nabla f(w_n), a-w_n \rangle - \langle \nabla f(z_{n+1}), a-z_{n+1} \rangle$$
$$\le f(z_{n+1}) - f(z_n) - (1-\alpha_n) f(w_n) \tag{3.2}$$

and

$$\alpha_n \langle \nabla f(z_n), b-z_n \rangle + (1-\alpha_n) \langle \nabla f(w_n), b-w_n \rangle - \langle \nabla f(z_{n+1}), b-z_{n+1} \rangle$$
$$\le f(z_{n+1}) - f(z_n) - (1-\alpha_n) f(w_n). \tag{3.3}$$

Multiply t and $(1-t)$ on both sides of (3.2) and (3.3) respectively, we get

$$\alpha_n \langle \nabla f(z_n), ta+(1-t)b-z_n \rangle + (1-\alpha_n) \langle \nabla f(w_n), ta+(1-t)b-w_n \rangle$$
$$-\langle \nabla f(z_{n+1}), ta+(1-t)b-z_{n+1} \rangle$$
$$\le f(z_{n+1}) - f(z_n) - (1-\alpha_n) f(w_n).$$

The above inequality yields

$$D_f(ta+(1-t)b, z_{n+1}) \le \alpha_n D_f(ta+(1-t)b, z_n))$$

$$+(1-\alpha_n) D_f(ta+(1-t)b, w_n) \Rightarrow ta+(1-t)b \in C_{n+1}$$ and hence C_{n+1} is closed and convex for all $n \ge 0$. Therefore, C_{n+1} is closed and convex subsets of E.

Step 3: we show that $\Omega \subset C_n$ for all $n \ge 0$, From the assumption that $C_0 = C$, we see that $\Omega \subset C_0 = C$. Suppose that $\Omega \subset C_n$ for all $n \ge 0$, since $T_j : C \to C, j = 1,2,3,\ldots,d$ is a finite family of continuous Bregman relatively nonexpansive mapping. Now for $q \in \Omega \subset C_n$, we obtain the following estimations;

$$D_f(q,z_{n+1}) = D_f(q, \nabla f^*(\beta_n \nabla f(u_n) + (1-\beta_n) \nabla f(\upsilon_n)))$$
$$\le \beta_n D_f(q,u_n) + (1-\beta_n) D_f(q,\upsilon_n)$$
$$= \beta_n D_f(q, T_{1,r_n}, y_n) + (1-\beta_n) D_f(q, T_{2,r_n}, y_n)$$

$$
\begin{aligned}
&\le D_f(q,u_n)\\
&= D_f(q,\nabla f^*(\alpha_n \nabla f(z_n)+(1-\alpha_n)\nabla f(T_j w_n)))\\
&\le \alpha_n D_f(q,z_n)+(1-\alpha_n)D_f(q,T_j w_n)\\
&\le \alpha_n D_f(q,z_n)+(1-\alpha_n)D_f(q,w_n) \qquad (3.4)
\end{aligned}
$$

that is $q\in C_{n+1}$. This implies by induction that $\Omega\subset C_n$ and the sequence generated (3.1) is well defined for all $n\ge 0$.

Step 4: We show that the sequences $\{x_n\},\{w_n\},\{z_n\}$ and $\{y_n\}$ are bounded. Since $x_n = proj^f_{C_n} x_0$ and $C_{n+1}\subset C_n$ for all $n\ge 0$ by Lemma 2.1, we obtain

$$D_f(x_{n+1},x_n)+D_f(x_n,x_0)\le D_f(x_{n+1},x_0)$$

Implies that

$$D_f(x_n,x_0)\le D_f(x_{n+1},x_0)$$

This shows that $\{D_f(x_n,x_0)\}$ is non-decreasing. Let $q\in\Omega$. It follows from Lemma 2.1 that

$$D_f(q,proj^f_{C_n}x_0)+D_f(proj^f_{C_n}x_0,x_0)\le D_f(q,x_0)$$

and so

$$D_f(x_n,x_0)\le D_f(q,x_0)-D_f(q,x_n)\le D_f(q,x_0),\forall n\ge 0.$$

Therefore, $\{D_f(x_n,x_0)\}$ is bounded. Consequently $\{D_f(x_n,x_0)\}$ is convergent. It follows that from Lemma 2.5 that $\{x_n\}$ is bounded. Furthermore, the inequality

$$D_f(q,x_n)=D_f(q,proj^f_{C_{n-1}\cap Q_{n-1}}x_0)\ \le D_f(q,x_0)-D_f(x_n,x_0),$$

this implies that $\{D_f(q,x_n)\}$ is bounded. Now by using the fact that $D_f(q,Tx_n)\le D_f(q,x_n),\forall q\in\Omega$, then $\{Tx_n\}$ is also bounded. Therefore $\{w_n\}$ and $\{y_n\}$ are also bounded. Setting $M=\max\{D_f(q,z_0),\sup_n D_f(q,w_n)\}$. Then obviously $D_f(q,z_0)\le M$. Let $D_f(q,z_n)\le M$ for some n, then it follows from (3.4) that

$$D_f(q,z_{n+1})\le \alpha_n M+(1-\alpha_n)M\le M$$

Thus, $\{D_f(q,z_{n+1})\}$ is bounded which implies that $\{z_n\}$ is bounded.

By using Lemma 2.1, we obtain

$$D_f(x_m, x_n) = D_f(x_{m,} proj_{C_n}^f x_0) \leq D_f(x_m, x_0) - D_f(x_n, x_0)$$

which gives

$$\lim_{n\to\infty} D_f(x_m, x_n) = 0,$$

and holds uniformly for all m. Since f is totally convex on bounded subsets of E, f is sequentially consistent then, it follows from Lemma 2.4 that

$$\lim_{n\to\infty} \|x_m - x_n\| = 0 \ .$$

This implies that the sequence $\{x_n\}$ is Cauchy. Therefore there exists a point $\hat{x} \in C$ such that $x_n \to \hat{x}$ (as $n \to \infty$). Also by using Lemma 2.1, we get

$$D_f(x_{n+1}, x_n) \leq D_f(x_{n+1}, x_0) - D_f(x_n, x_0) \,.$$

This implies

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0 \,.$$

Since f is totally convex on bounded subset of E, f is sequentially consistent, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.5}$$

Since ∇f is norm-to-norm uniformly continuous on bounded subsets of E, we get

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(x_n)\| = 0. \tag{3.6}$$

Now, by the definition of w_n from (3.1), we have

$$\nabla f(x_n) - \nabla f(w_n) = \nabla f(x_n) - \nabla f(x_n) + \theta_n (\nabla f(x_n) - \nabla f(x_{n-1})) \,.$$

Therefore,

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(w_n)\| = \|\theta_n (\nabla f(x_{n-1}) - \nabla f(x_n))\|$$

$$\leq \theta_n \|\nabla f(x_{n-1}) - \nabla f(x_n)\|$$

Using (3.6), implies that

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(w_n)\| = 0.$$

Since ∇f is norm-to-norm uniformly continuous on bounded subsets of E^*, we have

$$\lim_{n\to\infty} \|x_n - w_n\| = 0 \tag{3.7}$$

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

and so, $w_n \to \hat{x}$ as $n \to \infty$.

This also shows that w_n is bounded. Furthermore

$$\|x_{n+1} - w_n\| \le \|x_{n+1} - x_n\| + \|x_n - w_n\|.$$

Using (3.5) and (3.7), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - w_n\| = 0.$$

It follows from Lemma 2.2 that

$$\lim_{n\to\infty} D_f(x_{n+1}, w_n) = 0. \tag{3.8}$$

From the three point identity of the Bregman distance, we have

$$D_f(x_{n+1}, z_n) = \langle \nabla f(z_n) - \nabla f(x_{n+1}), q - x_{n+1} \rangle + D_f(q, z_n) - D_f(q, x_{n+1}).$$

Since f is bounded on bounded subset of E^*, then ∇f is bounded on bounded subsets of E^* and hence it follows from boundedness of $\{x_n\}, \{Tx_n\}$ and $\{z_n\}$ that the sequences $\{\nabla f(x_n)\}, \{\nabla f(Tx_n)\}$ and $\{\nabla f(z_n)\}$ are bounded in E^*, which implies that $\{D_f(x_{n+1}, z_n)\}$ is bounded. Since $x_{n+1} = proj^f_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$D_f(x_{n+1}, z_{n+1}) \le \alpha_n D_f(x_{n+1}, z_n) + (1-\alpha_n) D_f(x_{n+1}, w_n)$$

Using $\lim_{n\to\infty} \alpha_n = 0$ and (3.8), we obtain

$$\lim_{n\to\infty} D_f(x_{n+1}, z_{n+1}) = 0.$$

Since f is totally convex on bounded subset of E, f is sequentially consistent, we get

$$\lim_{n\to\infty} \|x_{n+1} - z_{n+1}\| = 0. \tag{3.9}$$

Also

$$\|x_n - z_{n+1}\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_{n+1}\|.$$

By (3.5) and (3.9), we get

$$\lim_{n\to\infty} \|x_n - z_{n+1}\| = 0. \tag{3.10}$$

So $z_{n+1} \to \hat{x}$ as $n \to \infty$.

Also this shows that z_{n+1} is bounded.

Taking into account that

$$\|w_n - z_{n+1}\| \le \|w_n - x_n\| + \|x_n - z_{n+1}\|.$$

Using (3.7) and (3.10), we obtain

$$\lim_{n\to\infty}\|w_n - z_{n+1}\| = 0. \tag{3.11}$$

Since f is uniformly Frechet differentiable, by using Lemma 2.10

$$\lim_{n\to\infty} | f(w_n) - f(z_{n+1}) | = 0, \tag{3.12}$$

and so

$$\lim_{n\to\infty}\|\nabla f(w_n) - \nabla f(z_{n+1})\| = 0. \tag{3.13}$$

By Bregman distance, we estimate as follows:

$$\begin{aligned} D_f(q,w_n) - D_f(q,z_{n+1}) &= f(q) - f(w_n) - \langle \nabla f(w_n), q - w_n\rangle \\ &\quad - \left(f(q) - f(z_{n+1}) - \langle \nabla f(z_{n+1}), q - z_{n+1}\rangle\right) \\ &= f(z_{n+1}) - f(w_n) + \langle \nabla f(z_{n+1}), q - z_{n+1}\rangle - \langle \nabla f(w_n), q - w_n\rangle \\ &= f(z_{n+1}) - f(w_n) + \langle \nabla f(z_{n+1}) - \nabla f(w_n), q - w_n\rangle \\ &\quad + \langle \nabla f(z_{n+1}), w_n - z_{n+1}\rangle \end{aligned} \tag{3.14}$$

Since $\{\nabla f(w_n)\}$ and $\{\nabla f(z_{n+1})\}$ are bounded, for each $q \in \Omega$. By (3.11), (3.12), (3.13) and (3.14), we obtain

$$\lim_{n\to\infty}(D_f(q,w_n) - D_f(q,z_{n+1})) = 0. \tag{3.15}$$

On the other hand, for each $q \in \Omega$ and $j = 1,2,\ldots,d$, by Lemma 2.11, we have

$$\begin{aligned} D_f(z_{n+1}, y_n) &\le D_f(q,y_n) - D_f(q,z_{n+1}) \\ &= D_f(q, \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(T_j w_n))) - D_f(q,z_{n+1}) \\ &\le \alpha_n D_f(q,z_n) + (1-\alpha_n) D_f(q,T_j w_n) - D_f(q,z_{n+1}) \\ &\le \alpha_n D_f(q,z_n) + (1-\alpha_n) D_f(q,w_n) - D_f(q,z_{n+1}) \\ &= \alpha_n (D_f(q,z_n) - D_f(q,w_n)) + D_f(q,w_n) - D_f(q,z_{n+1}) \end{aligned} \tag{3.16}$$

Since $\{D_f(q,w_n)\}$ and $\{D_f(q,z_{n+1})\}$ are bounded, $\lim_{n\to\infty}\alpha_n = 0$ using (3.15) in (3.16), we get

$$\lim_{n\to\infty} D_f(z_{n+1}, y_n) = 0.$$

Since f is totally convex on bounded subset of E, f is sequentially consistent, we have

$$\lim_{n\to\infty}\|z_{n+1} - y_n\| = 0, \tag{3.17}$$

and so $y_n \to \hat{x}$ as $n\to\infty$. This shows that $\{y_n\}$ is bounded.

Since ∇f is norm-to-norm uniformly continuous on bounded subsets of E, we have

$$\lim_{n\to\infty}\|\nabla f(z_{n+1}) - \nabla f(y_n)\| = 0, \tag{3.18}$$

Taking into account that

$$\|y_n - w_n\| \le \|y_n - z_{n+1}\| + \|z_{n+1} - w_n\|.$$

Using (3.17) and (3.11), we obtain

$$\lim_{n\to\infty}\|y_n - w_n\| = 0. \tag{3.19}$$

Also, since ∇f is norm-to-norm uniformly continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty}\|\nabla f(y_n) - \nabla f(w_n)\| = 0. \tag{3.20}$$

By the definition of y_n from (3.1), we have

$$\begin{aligned}\|\nabla f(y_n) - \nabla f(w_n)\| &= \|\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(T_j w_n) - \nabla f(w_n)\| \\ &= \|\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(w_n) - \nabla f(w_n) \\ &\quad + (1-\alpha_n)(\nabla f(T_j w_n) - \nabla f(w_n))\| \\ &= \|\alpha_n(\nabla f(z_n) + \nabla f(w_n)) + (1-\alpha_n)(\nabla f(T_j w_n) - \nabla f(w_n))\| \\ &\ge (1-\alpha_n)\|\nabla f(T_j w_n) - \nabla f(w_n)\| - \alpha_n\|\nabla f(z_n) - \nabla f(w_n)\|,\end{aligned}$$

implies that

$$(1-\alpha_n)\|\nabla f(T_j w_n) - \nabla f(w_n)\| \le \alpha_n\|\nabla f(z_n) - \nabla f(w_n)\| + \|\nabla f(y_n) - \nabla f(w_n)\|$$

It follows from $\lim_{n\to\infty}\alpha_n = 0$ and (3.20) that

$$\lim_{n\to\infty}\|\nabla f(T_j w_n) - \nabla f(w_n)\| = 0, \forall j = 1,2,3,\dots,d$$

Since f is norm-to-norm uniformly continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty}\|T_j w_n - w_n\| = 0, \forall j = 1,2,3,\dots,d \tag{3.21}$$

Step 5: We show that $\hat{x} \in \Omega$. First, we prove that $\hat{x} \in \bigcap_{j=1}^{d} F(T_j)$. Then it follows from the boundedness of the sequence $\{w_n\}$ and E is reflexive, that there exists a subsequence $\{w_{n_m}\}$ of $\{w_n\}$ such that $w_{n_m} \rightharpoonup \hat{x}$ as $m \to \infty$. Also, it follows from (3.7) that there exists a subsequence $\{w_{n_m}\}$ of $\{w_n\}$ such that $w_{n_m} \rightharpoonup \hat{x}$ as $m \to \infty$. Furthermore $\{w_n\}$ is Cauchy sequence, implies that $w_{n_m} \rightharpoonup \hat{x} \in C$ as $m \to \infty$. By using the fact that $w_{n_m} \rightharpoonup \hat{x} \in C$ as $m \to \infty$ and (3.21), we get

$$\lim_{n\to\infty}\|T_j w_{n_m} - w_{n_m}\| = 0, \forall j = 1,2,3,\dots,d \tag{3.22}$$

Since T_j is a finite family of continuous Bregman relatively nonexpansive, using (3.22), we obtain $\hat{x} \in F(T_j) = \hat{F}(T_j), \forall j = 1,2,3,\dots,d$. Therefore

$$\hat{x} \in \bigcap_{j=1}^{d} F(T_j)$$

Next, we show that $\hat{x} \in \bigcap_{k=1}^{2} GMEP(\Theta_k, \psi_k, \varphi_k) = F(T_{k,r}), k = 1,2$. Let $q \in \Omega$. First, we prove that $\|u_n - y_n\| = 0$. Now from $u_n = T_{1,r_n} y_n$, we have

$$\begin{aligned}
D_f(q,u_n) &= D_f(q, T_{1,r_n} y_n) \\
&\le D_f(q, y_n) \\
&= D_f(q, \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(T_j w_n))) \\
&\le \alpha_n D_f(q, z_n) + (1-\alpha_n) D_f(q, T_j w_n) \\
&\le \alpha_n D_f(q, z_n) + (1-\alpha_n) D_f(q, w_n) \\
&= \alpha_n D_f(q, z_n) + D_f(q, w_n) - \alpha_n D_f(q, w_n) \\
&= \alpha_n (D_f(q, z_n) - D_f(q, w_n)) + D_f(q, w_n),
\end{aligned}$$

implies that

$$D_f(q,u_n) - D_f(q,w_n) \le \alpha_n (D_f(q,z_n) - D_f(q,w_n))$$

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

Since $\lim_{n\to\infty}\alpha_n = 0$, we obtain

$$\lim_{n\to\infty}(D_f(q,u_n) - D_f(q,w_n)) = 0.$$

It follows that

$$\lim_{n\to\infty} D_f(w_n,u_n) = 0.$$

Since f is totally convex on bounded subset of E, f is sequentially consistent, we get

$$\lim_{n\to\infty}\|w_n - u_n\| = 0. \tag{3.23}$$

Taking into account that

$$\|u_n - y_n\| \le \|u_n - w_n\| + \|w_n - y_n\|.$$

Using (3.19) and (3.23), we have

$$\lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{3.24}$$

Since ∇f is norm-to-norm uniformly continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty}\|\nabla(u_n) - \nabla f(y_n)\| = 0.$$

From the assumption $r_n \subset [a,\infty)$ and $a > 0$, we get

$$\lim_{n\to\infty}\frac{\|\nabla(u_n) - \nabla f(y_n)\|}{r_n} = 0.$$

Also, since $\{y_n\}$ is bounded, there exist a subsequence $\{y_{n_m}\}$ of $\{y_n\}$ such that $y_{n_m} \rightharpoonup \hat{x} \in C$ as $m \to \infty$. It follows from (3.24) that there exists a subsequence $\{u_{n_m}\}$ of $\{u_n\}$ such that

$u_{n_m} \rightharpoonup \hat{x} \in C$ as $m \to \infty$. Now

$$\tau(u_n, y) + \frac{1}{r_n}\langle y - u_n, \nabla f(u_n) - \nabla f(y_n)\rangle \ge 0, \forall y \in C,$$

where

$$\tau(u_n, y) = \Theta_1(u_n, y) + \langle \psi_1 y_n, y - u_n\rangle + \varphi_1(y) - \varphi_1(u_n).$$

Replacing n by n_m and using (A_2), we get

$$\frac{1}{r_n}\langle y - u_{n_m}, \nabla f(u_{n_m}) - \nabla f(y_{n_m})\rangle \ge -\tau(u_{n_m}, y) \ge \tau(y, u_{n_m})$$

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

Letting $m \to \infty$, we obtain from $u_{n_m} \rightharpoonup \hat{x} \in C$ that

$$\tau(y, \hat{x}) \le 0, \forall y \in C,$$

for t with $0 < t \le 1$ and $y \in C$. Let $yt = ty + (1-t)\hat{x}$, since $y \in C$ and $\hat{x} \in C$, we have $y_t \in C$ and $\tau(y_t, \hat{x}), y \in C$. Now from ($A_1$) and ($A_3$), we have

$$\begin{aligned} 0 &= \tau(y_t, y_t) \\ &\le t\tau(y_t, y_t) + (1-t)\tau(y_t, \hat{x}) \\ &\le t\tau(y_t, y). \end{aligned}$$

Dividing by t, we get

$$\tau(y_t, y) \ge 0, \forall y \in C.$$

Letting $t \to 0$ and using (A_3), we have

$$\tau(\hat{x}, y) \ge 0, \forall y \in C.$$

This implies that $\hat{x} \in GMEP(\Theta_1, \psi_1, \varphi_1)$. Similarly from $u_n = T_{2,r_n} y_n$, by the same argument, we get $\hat{x} \in GMEP(\Theta_2, \psi_2, \varphi_2)$. Therefore $\hat{x} \in \bigcap_{k=1}^{2} GMEP(\Theta_k, \psi_k, \varphi_k)$.
Hence

$$\hat{x} \in \Omega = (\hat{x} \in \bigcap_{k=1}^{2} GMEP(\Theta_k, \psi_k, \varphi_k)) \bigcap (\bigcap_{j=1}^{d} F(T_j)).$$

Step 6: We show that $x_n \to \hat{x} = proj_{\Omega}^{f} x_0$. Let $\hat{u} = proj_{\Omega}^{f} x_0$. Since $\{x_n\}$ is weekly convergence, then it follows from $x_{n+1} = proj^{f} C_{n+1} x_0$ and $\hat{u} \in \Omega \subset C_{n+1}$ that

$$D_f(x_{n+1}, x_0) \le D_f(\hat{u}, x_0).$$

Then by Lemma 2.7, we have $x_n \to \hat{u}$ as $n \to \infty$, thus $\hat{x} = \hat{u}$. Hence the sequence $\{x_n\}$ converges strongly to a point $\hat{u} = proj_{\Omega}^{f} x_0$. Therefore, it follows from uniqueness of the limit that $\{x_n\}$ converges strongly to a point $\hat{x} = proj_{\Omega}^{f} x_0$. This completes the proof. ■

From Theorem (3.1), if $\varphi_1 = \varphi_2 = 0$, we have the following corollary;

Corollary 3.2. Let C be a nonempty closed and convex subset of a reflexive Banach space E and $f:E\to\mathbb{R}$ be coercive Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of E. Let $\Theta_1,\Theta_2 : C\times C\to\mathbb{R},\ k=1,2.$ be bi functions which satisfying assumptions (A1) – (A2), $\psi_1,\psi_2 : C\to E^*,\ k=1,2.$ be continuous monotone mappings. Let $T_j : E\to E, j=1,2,3,...,d$ be finite family of continuous Bregman relatively nonexpansive mappings, assume that

$$\Omega := \left(\bigcap_{k=1}^{2} GEP(\Theta_k,\psi_k)\right)\bigcap\left(\bigcap_{j=1}^{d} F(T_j)\right)\neq\phi .$$

Let $\{x_n\}$ and $\{z_n\}$ be a sequences generated by the iterative schemes.

$$\begin{cases} x_0, z_0\in C_0 = E \\ w_n = \nabla f^*\left(\nabla f(x_n)+\theta_n(\nabla f(x_n)-\nabla f(x_{n-1}))\right); \\ y_n = \nabla f^*(\alpha_n\nabla f(z_n)+(1-\alpha_n)\nabla f(T_j w_n)); \\ u_n\in C\ni\Theta_1(u_n,y)+\langle\psi_1 y_n, y-u_n\rangle+\dfrac{1}{r_n}\langle y-u_n,\nabla f(u_n)-\nabla f(y_n)\rangle; \\ v_n\in C\ni\Theta_2(\upsilon_n,y)+\langle\psi_2 y_n, y-\upsilon_n\rangle+\dfrac{1}{r_n}\langle y-\upsilon_n,\nabla f(\upsilon_n)-\nabla f(y_n)\rangle; \\ z_{n+1} = \nabla f^*(\beta_n\nabla f(u_n)+(1-\beta_n)\nabla f(\upsilon_n)); \\ C_{n+1} = \left\{z\in C : D_f(z,z_{n+1})\le\alpha_n D_f(z,z_n)+(1-\alpha_n)D_f(z,w_n)\right\}; \\ x_{n+1} = proj^f_{C_{n+1}} x_0, \forall n\ge 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$, such that $\lim_{n\to\infty}\alpha_n = 0$, $\{r_n\}\subset[a,\infty)$, for some $a>0$ and $\theta_n(x_n - x_{n-1})$ is the inertial term with $\theta_n\in(0,1)$. We shall define

$$T_{k,r}(x)=\left\{z\in C:\Theta_k(z,y)+\langle\psi_k x, y-z\rangle+\frac{1}{r}\langle y-z,\nabla f(z)-\nabla f(x)\rangle\ge 0,\forall y\in C\right\},\forall x\in E, k=1,2.$$

Then, $\{x_n\}$ converges strongly to $proj^f_\Omega x_0$, where $proj^f_\Omega x_0$ is the Bregman projection of C onto Ω.

From theorem (3.1), if $\psi_1=\psi_2=0$ and $\varphi_1=\varphi_2=0$, we obtain the following corollary;

Corollary 3.3. Let C be a nonempty closed and convex subset of a reflexive Banach space E and $f:E\to\mathbb{R}$ be coercive Legendre function which is bounded, uniformly Frechet

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

differentiable and totally convex on bounded subset of E. Let $\Theta_1, \Theta_2 : C \times C \to \mathbb{R}$, $k = 1,2$. be bi functions which satisfying assumptions (A$_1$) – (A$_4$). Let $T_j : E \to E, j = 1,2,3,...,d$ be finite family of continuous Bregman relatively nonexpansive mappings, assume that

$$\Omega := \left(\bigcap_{k=1}^{2} EP(\Theta_k)\right) \cap \left(\bigcap_{j=1}^{d} F(T_j)\right) \neq \phi .$$

Let $\{x_n\}$ and $\{z_n\}$ be a sequences generated by the iterative schemes.

$$\begin{cases} x_0, z_0 \in C_0 = E \\ w_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_n) - \nabla f(x_{n-1}))); \\ y_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(T_j w_n)); \\ u_n \in C \ni \Theta_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, \nabla f(u_n) - \nabla f(y_n)\rangle; \\ v_n \in C \ni \Theta_2(\upsilon_n, y) + \frac{1}{r_n}\langle y - \upsilon_n, \nabla f(\upsilon_n) - \nabla f(y_n)\rangle; \\ z_{n+1} = \nabla f^*(\beta_n \nabla f(u_n) + (1-\beta_n)\nabla f(\upsilon_n)); \\ C_{n+1} = \{z \in C : D_f(z, z_{n+1}) \le \alpha_n D_f(z, z_n) + (1-\alpha_n) D_f(z, w_n)\}; \\ x_{n+1} = proj_{C_{n+1}}^f x_0, \forall n \ge 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in [0,1], such that $\lim_{n\to\infty} \alpha_n = 0$, $\{r_n\} \subset [a, \infty)$, for some $a > 0$ and $\theta_n(x_n - x_{n-1})$ is the inertial term with $\theta_n \in (0,1)$. We shall define

$$T_{k,r}(x) = \left\{ z \in C : \Theta_k(z, y) + \frac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x)\rangle \ge 0, \forall y \in C \right\}, \forall x \in E, k = 1,2.$$

Then, $\{x_n\}$ converges strongly to $proj_\Omega^f x_0$, where $proj_\Omega^f x_0$ is the Bregman projection of C onto Ω.

From theorem (3.1), if $\Theta_1 = \Theta_2 = 0$ and $\varphi_1 = \varphi_2 = 0$, we have the following corollary;

Corollary 3.4. Let C be a nonempty closed and convex subset of a reflexive Banach space E and $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of E. Let $\psi_1, \psi_2 : C \to E^*$ be continuous monotone mappings. Let $T_j : E \to E, j = 1,2,3,...,d$ be finite family of continuous Bregman relatively nonexpansive mappings, assume that

$$\Omega := \left(\bigcap_{k=1}^{2} VIP(\psi_k, C)\right) \cap \left(\bigcap_{j=1}^{d} F(T_j)\right) \neq \phi .$$

Let $\{x_n\}$ and $\{z_n\}$ be a sequences generated by the iterative schemes.

$$\begin{cases} x_0, z_0 \in C_0 = E \\ w_n = \nabla f^*\left(\nabla f(x_n) + \theta_n(\nabla f(x_n) - \nabla f(x_{n-1}))\right); \\ y_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(T_j w_n)); \\ u_n \in C \ni \langle \psi_1 y_n, y - u_n \rangle + \dfrac{1}{r_n}\langle y - u_n, \nabla f(u_n) - \nabla f(y_n)\rangle; \\ v_n \in C \ni \langle \psi_1 y_n, y - \upsilon_n \rangle + \dfrac{1}{r_n}\langle y - \upsilon_n, \nabla f(\upsilon_n) - \nabla f(y_n)\rangle; \\ z_{n+1} = \nabla f^*(\beta_n \nabla f(u_n) + (1-\beta_n)\nabla f(\upsilon_n)); \\ C_{n+1} = \left\{z \in C : D_f(z, z_{n+1}) \leq \alpha_n D_f(z, z_n) + (1-\alpha_n) D_f(z, w_n)\right\}; \\ x_{n+1} = proj_{C_{n+1}}^f x_0, \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$, such that $\lim_{n\to\infty} \alpha_n = 0$, $\{r_n\} \subset [a, \infty)$, for some $a > 0$ and $\theta_n(x_n - x_{n-1})$ is the inertial term with $\theta_n \in (0,1)$. We shall define $T_{k,r}(x) = \left\{z \in C : \langle \psi_k y_n, y - u_n \rangle + \dfrac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x)\rangle \geq 0, \forall y \in C\right\}, \forall x \in E, k = 1,2.$

Then, $\{x_n\}$ converges strongly to $proj_{\Omega}^f x_0$, where $proj_{\Omega}^f x_0$ is the Bregman projection of C onto Ω.

Corollary 3.5. Let C be a nonempty closed and convex subset of a reflexive Banach space E. Let $\Theta_1, \Theta_2 : C \times C \to \mathbb{R}$, $k = 1,2.$ be bi functions which satisfying assumptions (A_1) – (A_4), $\psi_1, \psi_2 : C \to E^*$, $k = 1,2.$ be continuous monotone mappings and $\varphi_1, \varphi_2 : C \to \mathbb{R}$, $k = 1,2.$ be convex and semi-continuous functions. Let $T_j : E \to E, j = 1,2,3,...,d$ be finite family of continuous relatively nonexpansive mappings, assume that

$$\Omega := \left(\bigcap_{k=1}^{2} GMEP(\Theta_k, \psi_k, \varphi_k)\right) \cap \left(\bigcap_{j=1}^{d} F(T_j)\right) \neq \phi .$$

Let $\{x_n\}$ and $\{z_n\}$ be a sequences generated by the iterative schemes.

$$\begin{cases} x_0, z_0 \in C_0 = E \\ w_n = x_n + \theta_n (x_n - x_{n-1}); \\ y_n = J^{-1}(\alpha_n J z_n + (1-\alpha_n) J T_j w_n); \\ u_n \in C \ni \Theta_1(u_n, y) + \langle \psi_1 y_n, y - u_n \rangle + \varphi_1(y) - \varphi_1(u_n) \\ + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle; \\ \upsilon_n \in C \ni \Theta_2(\upsilon_n, y) + \langle \psi_2 y_n, y - \upsilon_n \rangle + \varphi_2(y) - \varphi_2(\upsilon_n) \\ + \frac{1}{r_n} \langle y - \upsilon_n, J \upsilon_n - J y_n \rangle; \\ z_{n+1} = J^{-1}(\beta_n J u_n + (1-\beta_n) J \upsilon_n); \\ C_{n+1} = \{ z \in C : \phi(z, z_{n+1}) \le \alpha_n \phi(z, z_n) + (1-\alpha_n) \phi(z, w_n) \}; \\ x_{n+1} = \prod_{C_{n+1}} x_0, \forall n \ge 0, \end{cases}$$

where j is normalized duality mapping, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$, such that $\lim_{n\to\infty} \alpha_n = 0$, $\{r_n\} \subset [a, \infty)$, for some $a > 0$ and $\theta_n (x_n - x_{n-1})$ is the inertial term with $\theta_n \in (0,1)$. We shall define $T_{k,r}(x) = \left\{ z \in C : \Theta_k(z, y) + \langle \psi_k x, y - z \rangle + \varphi_k(y) - \varphi_k(z) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \ge 0, \forall y \in C \right\}$,

$\forall x \in E,\ k = 1,2.$

Then, $\{x_n\}$ converges strongly to $\prod_\Omega x_0$ is the projection of C onto Ω.

Applications

In this section, we present some applications of theorem 3.1 as follows:

4.1. Finite family of continuous Bregman relatively nonexpansive mappings and system of equilibrium problems. By setting $\psi \equiv 0$, $\varphi \equiv 0$ in theorem 3.1, the sequence $\{x_n\}$ defined in 3.1 converges strongly to $proj^f_\Omega x_0$ where $\Omega := \left(\bigcap_{k=1}^2 EP(\Theta_k)\right) \bigcap \left(\bigcap_{j=1}^d F(T_j)\right) \ne \phi$ and $EP(\Theta)$ is the solutions of the equilibrium problem for Θ.

4.2. Finite family of continuous Bregman relatively nonexpansive mappings and system of convex optimization problems. By setting $\Theta \equiv 0$, $\psi \equiv 0$ in theorem 3.1, the sequence $\{x_n\}$

defined in 3.1 converges strongly to $proj_{\Omega}^{f}x_0$ where $\Omega := \left(\bigcap_{k=1}^{2} GMP(\psi_k)\right) \cap \left(\bigcap_{j=1}^{d} F(T_j)\right) \neq \phi$ and $CMP(\varphi)$ is the solutions of the convex optimization problem for φ.

4.3. Finite family of continuous Bregman relatively nonexpansive mappings and system of variational inequalities problems. By setting $\Theta \equiv 0$, $\varphi \equiv 0$ in theorem 3.1, the sequence $\{x_n\}$ defined in 3.1 converges strongly to $proj_{\Omega}^{f}x_0$ where $\Omega := \left(\bigcap_{k=1}^{2} VIP(\psi_k, C)\right) \cap \left(\bigcap_{j=1}^{d} F(T_j)\right) \neq \phi$ and $VIP(C,\psi)$ is the solutions of the variational inequalities problem for ψ.

r

References:

Agha, O. I., Ibiam, L. O., Madu, E. U., Ofoedu, C. E., Onyi and H. Zegeye. (2017). Hybrid Algorithm for Nonlinear Equilibrium, Variational Inequality and Fixed Point Problems. *Journal of Physical Research. 7(1); ISSN; 2141 – 8403.*

Alansari, M., Ali, R and Farid, M. (2020). Strong Convergence of an Inertial Iterative Algorithm for Variational Inequality Problem, Generalised Equilibrium Problem and Fixed Point Problem in a Banach Space. *Journal of Inequalities and Applications. 42; 1 - 22.*

Alber, Y. I. (1996). Metric and Generalized Projection Operators in Banach Spaces: Properties and Applications, in: A. G. Kartsatos (Ed). *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Mercel Dekker, New York.*

Ali, B., Ezeora, J. N and Lawan M. S. (2019). Inertial Algorithm for Solving Fixed Point and Generalized Mixed Equilibrium Problems in Banach Space. *Pan American Mathematical Journal. 29(3); 64 – 83.*

Ali, B and Harbau, M. H. (2016). Convergence Theorem for Bregman K-Mappings and Mixed Equilibrium Problems in Reflexive Banach Spaces. *Journal of Function Space. ID 5161682.*

Ali, B., Ugwunnadi, G. C., Idris, I and Minjibir, M. S. (2014). Strong Convergence Theorem for Quasi Bregman Strictly Pseudocontractive, Mappings and Equilibrium Problems in Banach Spaces. *Fixed Point Theory and Applications. 231; 1 – 17. Doi :10.1186/1687-1812.*

Asplund, E and Rockafellar, R. T. (1969). Gradient of Convex Function, *Trans. Am. Math. Soc. 228; 443 – 467.*

Bauschke, H. H and Borwein, J. M. (1997). Legendre Function and the Method of Bregman Projections. *Journal of Convex Analysis. 4; 27 – 67.*

Bauschke, H. H., Combettes, P. L and Borwein, J. M. (2001). Essential Smoothness , Essential Strict Convexity and Legendre Functions in Banach Spaces. *Commun. Contemp. Math. 3; 615 – 647.*

Bonnans, J. F and Shapiro, A. (2000). Perturbation Analysis of Optimization Problems. *Springer New York.*

Bot, R. I., Csetnek, E. R and Hendrich, C. (2015). Inertial Douglas-Racheord Splitting for Monotone Inclusion Problems. *Appl. Math. Comput. 256; 472 – 478.*

Bregman, L. M. (1967). The Relaxation Method for Finding the Common Point of Convex Set and its Application to Solution of Convex Programming. *USSR Comput. Math. Phys.7; 200 – 217.*

Butnariu, D and Lusem, A. N. (2000). Totally Convex Function for Fixed Points Computation and Infinite Dimensional Optimization. *Kluwer Academic Dordrecht.*

Butnariu, D and Resmerita, E. (2006). Bregman Distance, Totally Convex Functions and a Method for Solving Operator Equations in Banach Spaces. *Abstr. Anal. Article ID:84919.*

Blum, E and Oettli, W. (1994). From Optimization and Variational Inequalities to Equilibrium Problems. *Math. Stud. 63; 123 – 145.*

Bot, R. I and Csetnek, E. R. (2016). An Inertial Forward-Backward-Forward Primal-Dual Splitting Algorithm for solving Monotone Inclusions Problems. *Numer Algorithms. 71; 519 – 540.*

Censor, Y and Lent, A. (1981). An Iterative Row-Action Method for Interval Convex Programming. *Journal of Optimization Theory and Application. 34; 321 – 353.*

Chang, S. S., Wang, L., Wang, X, R and Chan, C. K. (2013). Strong Convergence Theorems for Bregman Totally Quasi-asymptotical Nonexpansive Mappings in Reflexive Banach Spaces. *Appl. Math. Comput. Doi:101016/jamc2013,11.074.*

Chidume, C. E., Ikechukwu, S. I and Adamu, A. (2018). Inertial Algorithm for Approximating a Common Fixed Point for Countable Family of Relatively Nonexpansive Mappings. *Fixed Point Theory and Applications. 9; 1 – 9.*

Combeltes, P. L and Hirstoaga, S. A. (2005). Equilibrium Programming in Hilbert Spaces. *Nonlinear Convex Analysis. 6; 117 – 136.*

Darvish, V. (2016). Strong Convergence Theorem for a System of Generalized Mixed Equilibrium Problems and Finite Family of Bregman Nonexpansive Mappings in Banach Spaces. *Opsearch. 53; 584 – 603. DOI10.1007/s12597 – 015 – 1245 – 2.*

Darvish, V., Jantakawa, A. Kaewcharoen, A and Biranvand, N. (2019). Strong Convergence Theorem for Solving a Generalized Mixed Equilibrium Problems and Finding Fixed Point of a Weak Bregman Relatively Nonexpansive Mappings in Banach Spaces. Functional Analysis arXiv: 1911.02246(Math), Cornell University.

Dong, O. L., Yuan, H. B., Je, C. Y and Rassias, T. M. (2018). Modified Inertial Algorithm and Inertial CQ Algorithm for Nonexpansive mappings. *Optim. Lett. 12; 87 – 102.*

Hiriart-Urruty and Lemarchal, J. B. (1993). Convex Analysis and Minimization Algorithm. Grudlehren der Mathematischen Wissenchaften, 306, Springer, Berlin.

Jantakan, K and Kaewchroen, A. (2021). Strong Convergence Theorems for Mixed Equilibrium Problems and Bregman Relatively Nonexpansive Mappings in Reflexive Banach Spaces with Application. *Journal Nonlinear Science and Application. 14; 63 – 79.*

Kazmi, K. R., Ali, R and Yousuf, S. (2018). Generalized Equilibrium and Fixed point Problems for Bregman Relatively Nonexpansive Mappings in Banach Spaces. *Journal of Fixed Point Theory and Applications. 20; 151. https://doi.org/10.1007/s11784 - 018-0627-1.*

Lorenz, D and Pock, T. (2015). An Inertial Forward-Backward Algorithm for Monotone Inclusions. *Journal of Mathematics Imaging. 51; 311 – 325.*

Martin-Marquez, Reich, V and Sabach, S. (2013). Iterative Method for Approximating Fixed Points of Bregman Nonexpansive Operators. *Discrete Contin, Dyn. Syst. 6; 1043 – 1063.*

Naraghired, E and Yao, J. C. (2013). Bregman Weak Relatively Nonexpansive Mappings in Banach Spaces. *Fixed Point Theory and Applications. 1 - 43. Doi:10.1186/1687 – 1812 – 2013 – 141.*

Polyak, B. T. (1964). Some Methods of Speeding up the Convergence of Iteration Methods. *USSR Comput. Math. Phys. 4(5); 1 – 7.*

Reich, S and Sabah, S. (2009). A Strong Convergence Theorem for a Proximal-Type Algorithm in reflexive Banach Spaces. *Journal of Nonlinear Convex Analysis. 10; 471 – 485.*

Reich, S and Sabah, S. (2010). Two Strong Convergence Theorems for a Proximal Method in Reflexive Banach spaces. *Numer. Funct. Anal. Optim. 31; 22 – 44.*

Reich, S and Sabah, S. (2011). A Projection Method for Solving Nonlinear Problems in Banach spaces. *Journal of Fixed Point Theory and Application. 9; 101 – 116.*

Takahashi, W and Zembayashi, K. (2009). Strong and Weak Convergence Theorems for Equilibrium Problems and relatively Nonexpansive Mappings in Banach Spaces. *Nonlinear Analysis. 70; 45 – 57.*

Zalinescu, C. (2002). Convex Analysis in General vector Spaces. World Scientific, River Edge.

Zhang, M and Cho, S. Y. (2016). A Monotone Projection Algorithm for Solving Fixed Points of Nonlinear Mappings and Equilibrium Problems. *Journal of Nonlinear Sciences and Applications. 9; 1453 – 1562.*

CONTINUOUS FORMULATION OF HYBRID BLOCK MILNE TECHNIQUE FOR SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

K. J. Audu[1*], Y. A. Yahaya[2], J. Garba[3], A. T. Cole[4] and F. U. Tafida[5]
[,1,2,3,4]Department of Mathematics, Federal University of Technology, P.M.B. 65 Minna, Nigeria
[5]Department of Mathematics, Ibrahim Badamasi Babangida University, P.M.B. 11, Lapai, Nigeria
[*]Corresponding author: email: k.james@futminna.edu.ng

Abstract

In most scientific and engineering problems, ordinary differential equations cannot be solved by analytic methods. Consequently, numerical approaches are frequently required. A block hybrid Milne technique was formulated in this paper in order to develop a suitable algorithm for the numerical solution of ordinary differential equations. Utilizing power series as the basis function, the proposed method is developed. The developed algorithm is used to solve systems of linear and nonlinear differential equations, and it has proven to be an efficient numerical method for avoiding time-consuming computation and simplifying differential equations. The fundamental numerical properties are examined, and the results demonstrate that it is zero-stable and consistent, which ensures convergence. In addition, by comparing the approximate solutions to the exact solutions, we demonstrate that the approximate solutions converge to the exact solutions. The results demonstrate that the developed algorithm for solving systems of ordinary differential equations is straightforward, efficient, and faster than the analytical method.

Keywords: Ordinary differential equations, numerical solution of ODEs, Hybrid Milne method, approximate solutions, algorithm and power series

Introduction

An equation in mathematics that describes the relationship between a function and its derivative is an example of a differential equation. In practical contexts, functions are typically used to represent rates of change. Engineers, physicists, economists, biologists, and others rely heavily on differential equations. Initial value first order ordinary differential equations appear in the process of modeling real-world situations in physical and applied sciences, particularly in algebraic expressions concerning problems related to flow of viscous thin films, disease models, chemical kinetics, quantum mechanics and electromagnetic waves (Aslam *et al.*, 2021; Mazarina and Syahirbanun (2022); Amat *et al.*, 2019; Kwanamu *et al.*, 2021). Understanding the behaviors and propertiesof the investigated physical phenomena requires the resolution of this type of problem (Kashkaria and Syam (2019)). In the majority of instances, available analytical approaches fail to provide an accurate solution to a general first-order initial value problem. To solve such problems that come up in various area of engineering and science, it is important to use numerical approaches that are close to the equations' solutions (Chapra and Canale, 2015). As such, scientific and technological problems involving differential equations are typically solved using numerical methods rather than analytic ones.

In this research, we intend to develop and study a four-step first derivative hybrid block Milne approach for systems of ordinary differential equations taken into account as:

$$\begin{cases} z_1' = f(t, z_1, z_2, \ldots \\ z_2' = f(t, z_1, z_2, \ldots \\ \vdots \\ z_n' = f(t, z_1, z_2, \ldots \end{cases} \quad \text{with initial conditions} \quad \begin{cases} z_1(0) = c_1 \\ z_2(0) = c_2 \\ \vdots \\ z_n(0) = c_n \end{cases} \qquad (1)$$

For arbitrary $z_0 \le z \le z_N$. In this case, the function $f(t,z)$ is assumed to be continuous throughout the integration interval, and a unique solution exists. Numerous research has been carried out to provide numerical solutions to problems modeled as first order ordinary differential equations. These include works of authors such as Ndipmong and Udechukwu (2022), Garba and Mohammed (2020), Gomathi and Rabiyabe (2022), Badmus *et al.* (2015), Ehiemua and Agbeboh (2019), Eziokwu and Okereke (2020). Iyorter, B. V., Luga, T. & Isah, S. S. (2019). Techniques of solution employed by the above researchers include, Euler methods, the Adams Bashforth and Adams Moulton methods, linear multistep methods, Runge-Kutta methods and Milne methods among others.

Few mathematicians have come up with some block Milne techniques regarding solutions to various differential problems. The convergence of some selected properties with respect to block predictor-corrector methods and its applications on differential problems were investigated (Oghonyon *et al.*, 2016a). Again, Oghonyon *et al.,* (2016b) focused on block predictor-corrector method and derived a Milne's scheme. They implemented the scheme on ordinary differential problems and obtained a favourable outcome. Recently, Oghonyon *et al*. (2018a) formulated a suitable exponential fitted block Milne's scheme for ordinary differential equations emerging from oscillating vibrations problems. However, these approaches are limited by their low accuracy rate and low number of steps. The present research was motivated by the need to overcome the shortcomings of existing approaches by expanding the number of steps at both grid and off-step locations. The Milne technique employs the predictor-corrector algorithm and is dahlquist stable and accurate to the second order. For their starting values, the predictor-corrector of the Milne scheme requires single-step methods. In this study, the corrector component is reformulated into a continuous form and implemented as a block method in order to make it self-starting to solve systems of ordinary differential equations. To improve the degree of accuracy of the Milne method, appropriate off-grid points are selected with care.

This paper is structured as follows. In Section two, we describe the construction of the new numerical technique for (1). In section three, we established the order, zero stability, consistency, and convergence of the technique. In Section four, we used the method to solve systems of differential equations of the first order and compared the results of the different problems. Numerical tests with sample problems and their results were resented in section 4 and we concluded the study in section 5.

Construction of the Block Hybrid Milne Technique

To derive the new numerical technique, we apply the notion of a linear multistep collocation procedure using the general format $z(t) = \sum_{n=0}^{s+1} A(t) z(t_{i+n}) = h \sum_{n=0}^{v-1} B(t) f(\overline{t}_n, \overline{z}(\overline{t}_n))$

(2)

where

$$A_i(t)=\sum_{n=0}^{u+v-1} A_{i,n+i}t^n \quad and \quad hB_i(t)=\sum_{n=0}^{w+v-1} B_{i,n+i}t^n \tag{3}$$

Here, we use the basis function of power series to derive a numerical estimate for the ordinary differential equation of the format described in (1).

$$\sum_{n=0}^{w+v-1} d_n t^n \tag{4}$$

where w and v represents the interpolation and collocation points, $t \in [t_0, z_N]$, and d_n's are unknowns. Equation (1) is differentiated to get

$$\sum_{n=0}^{w+v-1} n d_n t^{n-1} \tag{5}$$

Hence, the continuous format of the proposed block technique from (3) with five off grid points at collocation is represented as

$$\begin{aligned} z(t) = {} & A_2(t)z_{i+2} + B_2(t)hf_{i+2} + B_{\frac{9}{4}}(t)hf_{i+\frac{9}{4}} + B_{\frac{5}{2}}(t)hf_{i+\frac{5}{2}} + B_3(t)hf_{i+3} + B_{\frac{13}{4}}(t)hf_{i+\frac{13}{4}} \\ & + B_{\frac{7}{2}}(t)hf_{i+\frac{7}{2}} + B_4(t)hf_{i+4} \end{aligned} \tag{6}$$

It generated some non-linear system of equations in the format $Mt = B$ in (7)

$$\begin{pmatrix} 1 & (t_{i+2}) & (t_{i+2})^2 & (t_{i+2})^3 & (t_{i+2})^4 & (t_{i+2})^5 & (t_{i+2})^6 & (t_{i+2})^7 \\ 0 & 1 & 2(t_{i+2}) & 3(t_{i+2})^2 & 4(t_{i+2})^3 & 5(t_{i+2})^4 & 6(t_{i+2})^5 & 7(t_{i+\frac{9}{4}})^6 \\ 0 & 1 & 2(t_{i+\frac{9}{4}}) & 3(t_{i+\frac{9}{4}})^2 & 4(t_{i+\frac{9}{4}})^3 & 5(t_{i+\frac{9}{4}})^4 & 6(t_{i+\frac{9}{4}})^5 & 7(t_{i+\frac{9}{4}})^6 \\ 0 & 1 & 2(t_{i+\frac{5}{2}}) & 3(t_{i+\frac{5}{2}})^2 & 4(t_{i+\frac{5}{2}})^3 & 5(t_{i+\frac{5}{2}})^4 & 6(t_{i+\frac{5}{2}})^5 & 7(t_{i+\frac{5}{2}})^6 \\ 0 & 1 & 2(t_{i+3}) & 3(t_{i+3})^2 & 4(t_{i+3})^3 & 5(t_{i+3})^4 & 6(t_{i+3})^5 & 7(t_{i+3})^6 \\ 0 & 1 & 2(t_{i+\frac{13}{4}}) & 3(t_{i+\frac{13}{4}})^2 & 4(t_{i+\frac{13}{4}})^3 & 5(t_{i+\frac{13}{4}})^4 & 6(t_{i+\frac{13}{4}})^5 & 7(t_{i+\frac{13}{4}})^6 \\ 0 & 1 & 2(t_{i+\frac{7}{2}}) & 3(t_{i+\frac{7}{2}})^2 & 4(t_{i+\frac{7}{2}})^3 & 5(t_{i+\frac{7}{2}})^4 & 6(t_{i+\frac{7}{2}})^5 & 7(t_{i+\frac{7}{2}})^6 \\ 0 & 1 & 2(t_{i+4}) & 3(t_{i+4})^2 & 4(t_{i+4})^3 & 5(t_{i+4})^4 & 6(t_{i+4})^5 & 7(t_{i+4})^6 \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \end{pmatrix} = \begin{pmatrix} z_{i+2} \\ f_{i+2} \\ f_{i+\frac{9}{4}} \\ f_{i+\frac{5}{2}} \\ f_{i+3} \\ f_{i+\frac{13}{4}} \\ f_{i+\frac{7}{2}} \\ f_{i+4} \end{pmatrix} \tag{7}$$

Employing Maple 2015 software to compute (7), and evaluation of the desired points results into the following proposed schemes;

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

$$z_{i+1} = -\frac{45373}{1260}hf_{i+2} + \frac{48518}{315}hf_{i+3} - \frac{3292832}{945}hf_{i+4} - \frac{134941}{945}hf_{i+\frac{5}{2}} + \frac{3035}{63}hf_{i+\frac{7}{2}} + \frac{269296}{2205}hf_{i+\frac{9}{4}} - \frac{27248}{189}hf_{i+\frac{13}{4}} + z_{i+2} \quad (8)$$

$$z_i = z_{i+2} - \frac{72353}{105}hf_{i+2} + \frac{1994752}{735}hf_{i+\frac{9}{4}} - \frac{3292832}{945}hf_{i+\frac{5}{2}} + \frac{443852}{105}hf_{i+3} - \frac{778240}{189}hf_{i+\frac{13}{4}} + \frac{149216}{105}hf_{i+\frac{7}{2}} - \frac{482819}{6615}hf_{i+4} \quad (9)$$

$$z_{i+\frac{9}{4}} = \frac{2251}{26880}hf_{i+2} + \frac{1661}{26880}hf_{i+3} - \frac{293}{423360}hf_{i+4} - \frac{23021}{241920}hf_{i+\frac{5}{2}} + \frac{85}{5376}hf_{i+\frac{7}{2}} + \frac{5549}{23520}hf_{i+\frac{9}{4}} - \frac{311}{6048}hf_{i+\frac{13}{4}} + z_{i+2} \quad (10)$$

$$z_{i+\frac{5}{2}} = \frac{787}{10080}hf_{i+2} + \frac{73}{2520}hf_{i+3} - \frac{83}{211680}hf_{i+4} + \frac{863}{15120}hf_{i+\frac{5}{2}} + \frac{43}{5040}hf_{i+\frac{7}{2}} + \frac{781}{2205}hf_{i+\frac{9}{4}} - \frac{5}{189}hf_{i+\frac{13}{4}} + z_{i+2} \quad (11)$$

$$z_{i+3} = \frac{37}{420}hf_{i+2} + \frac{44}{105}hf_{i+3} - \frac{47}{26460}hf_{i+4} + \frac{331}{945}hf_{i+\frac{5}{2}} + \frac{1}{21}hf_{i+\frac{7}{2}} + \frac{208}{735}hf_{i+\frac{9}{4}} - \frac{176}{945}hf_{i+\frac{13}{4}} + z_{i+2} \quad (12)$$

$$z_{i+\frac{13}{4}} = \frac{1405}{16128}hf_{i+2} + \frac{8875}{16128}hf_{i+3} - \frac{125}{84672}hf_{i+4} + \frac{16375}{48384}hf_{i+\frac{5}{2}} + \frac{575}{16128}hf_{i+\frac{7}{2}} + \frac{4075}{14112}hf_{i+\frac{9}{4}} - \frac{295}{6048}hf_{i+\frac{13}{4}} + z_{i+2} \quad (13)$$

$$z_{i+\frac{7}{2}} = \frac{99}{1120}hf_{i+2} + \frac{141}{280}hf_{i+3} - \frac{17}{7840}hf_{i+4} + \frac{197}{560}hf_{i+\frac{5}{2}} + \frac{15}{112}hf_{i+\frac{7}{2}} + \frac{69}{245}hf_{i+\frac{9}{4}} + \frac{1}{7}hf_{i+\frac{13}{4}} + z_{i+2} \quad (14)$$

$$z_{i+4} = \frac{17}{315}hf_{i+2} + \frac{404}{315}hf_{i+3} - \frac{869}{6615}hf_{i+4} + \frac{32}{945}hf_{i+\frac{5}{2}} + \frac{352}{315}hf_{i+\frac{7}{2}} + \frac{1024}{2205}hf_{i+\frac{9}{4}} - \frac{1024}{945}hf_{i+\frac{13}{4}} + z_{i+2} \quad (15)$$

Analysis of the Proposed Technique

This section is concerned on analyses with respect to zero stability and consistency of the novel technique.

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

Consistency

The proposed technique described in section 2 is frequently written as;

$$\sum_{i=0}^{4} \mathrm{A}_i z_{n+i} - \sum_{i=0}^{4} h\mathrm{B} f_{n+i} = 0 \qquad (16)$$

Following Oghonyon *et al.* (2018b) and Mohammed *et al.* (2021), the local truncation error is a linear difference operator as;

$$L[z(t);h] = z_i - h\left(\mathrm{B}_2(t)z'_n + \mathrm{B}_3(t)z'_n + \mathrm{B}_{\frac{5}{2}}(t)z'_n + \mathrm{B}_{\frac{7}{2}}(t)z'_n + \mathrm{B}_{\frac{9}{4}}(t)z'_n + \mathrm{B}_{\frac{13}{4}}(t)z'_n + \mathrm{B}_4(t)z'_n\right) \qquad (17)$$

Assuming that $z(t)$ is sufficiently differentiable, then the Taylor's expansion of (17) about the point t, can be represented as;

$$L[z(t);h] = E_0 z(t) + E_1 hz'(t) + E_2 h^2 z''(t) + \cdots + E_p h^p z^p(t) + E_{p+1} h^{p+1} z^{p+1}(t) \quad (18)$$

The discrete scheme in (9) is said to be consistent if $p \geq 1$ for $E_0 = E_1 = E_2 = \cdots = E_p = 0, E_{p+1} \neq 0$, where E_{p+1} denotes the error constant, and p denotes the order of the hybrid technique (Tiamiyu *et al.,* 2021). The summary of the order and error constant of the block schemes is given in Table 1.

Table 1 –Error Constants and Order of the Proposed Technique

Equation	Order	Error constant
(8)	7	$\frac{1643}{15680}$
(9)	7	$\frac{70099}{27095040}$
(10)	7	$\frac{1051}{2055208960}$
(11)	7	$\frac{9925}{11098128384}$
(12)	7	$\frac{1}{1003520}$
(13)	7	$\frac{139}{433520640}$
(14)	7	$\frac{17}{16056320}$
(15)	7	$-\frac{1}{211680}$

Zero Stability

To determine the zero stability of the new derived schemes, the first characteristic polynomial $R(\lambda)$ of (8) to (15) denoted as $\det(\lambda \times A(1) - A(0))$ is normalized as follows;

$R(\lambda) = \det(\lambda \times A(1) - A(0))$ such that we obtain

$$R(\lambda) = \lambda \times \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = (\lambda^2 - \lambda)\lambda^6$$

for $|\lambda| \le 1$ and the roots $|\lambda| = 1$, the multiplicity must not exceed one. Hence, we arrive at the deduction $R(\lambda) = \det(\lambda \times A(1) - A(0)) = (\lambda^2 - \lambda)\lambda^6 = 0$ and $\lambda = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1)$. Therefore, the developed hybrid block Milne technique is said to be zero stable.

Convergence

According to Ma'ali *et al.* (2020), Dahlquist's fundamental theorem asserts that "the necessary and sufficient requirements for a linear multi-step procedure to be convergent are consistency and zero-stability. By Kashkaria and Syam (2019) and Oghonyon *et al*. (2018b), since the hybrid block approach provided is consistent and zero stable, the convergence requirement is met.

Numerical Tests

Problem 1: We consider a set of linear differential equations in the form;

$$\begin{aligned} z_1' &= -21z_1 + 19z_2 - 20z_3, & z_1(0) &= 1 \\ z_2' &= 19z_1 - 21z_2 + 20z_3, & z_2(0) &= 0 \\ z_3' &= 40z_1 - 40z_2 - 40z_3, & z_3(0) &= -1 \\ 0 &\le t \le 3, & h &= 0.2 \end{aligned}$$

The exact solution is provided as

$$z_1(t)=\frac{1}{2}e^{-2t}+\frac{1}{2}e^{-40t}\sin(40t)+\frac{1}{2}e^{-40t}\cos(40t)$$
$$z_2(t)=\frac{1}{2}e^{-2}-\frac{1}{2}e^{-40t}\sin(40t)-\frac{1}{2}e^{-40t}\cos(40t)$$
$$z_3(t)=e^{-40t}\sin(40t)-e^{-40t}\cos(40t)$$

Problem 2: Considering the systems of initial value problem of first order differential equation of the form;

$$z_1' = -z_1 + 95z_2, \qquad z_1(0)=1$$
$$z_2' = -z_1 - 97z_2, \qquad z_2(0)=1$$
$$h = 0.0625$$

The real solution is provided as;

$$z_1(t)=\frac{95}{47}e^{-2t}-\frac{48}{47}e^{-96t}$$
$$z_2(t)=\frac{48}{47}e^{-96t}-\frac{1}{47}e^{-2t}$$

Problem 3: We consider the systems of initial value problem of first order differential equation of the form;

$$z_1' = -(2+10^4)z_1 + 10^4 z_2, \qquad z_1(0)=1$$
$$z_2' = z_1 - z_2 - z_2^2, \qquad z_2(0)=1$$

With $h=0.1$ and the exact solution given as

$$z_1(t)=e^{-2t}$$
$$z_2(t)=e^{-t}$$

Problem 4. Solving the non-linear system of initial value problem of first order differential equation of the form;

$$z_1' = -1002z_1 + 100z_2^{\,2}, \qquad z_1(0)=1$$
$$z_2' = z_1 - z_2(1+z_2), \qquad z_2(0)=1$$

$0 \le t \le 1$ and the exact solution can be obtained from the following relations

$$z_1(t)=e^{-2t},\; z_2(t)=e^{-t}$$

Test Results

This section presents the test results for problems 1 to 4 considered in previous section. Comparison of the computations are displayed in some Figures and Tables. The exact solutions are represented by $z(t)$ and the new hybrid Block Milne solutions are denoted as $z_s(t), s = 1, 2$.

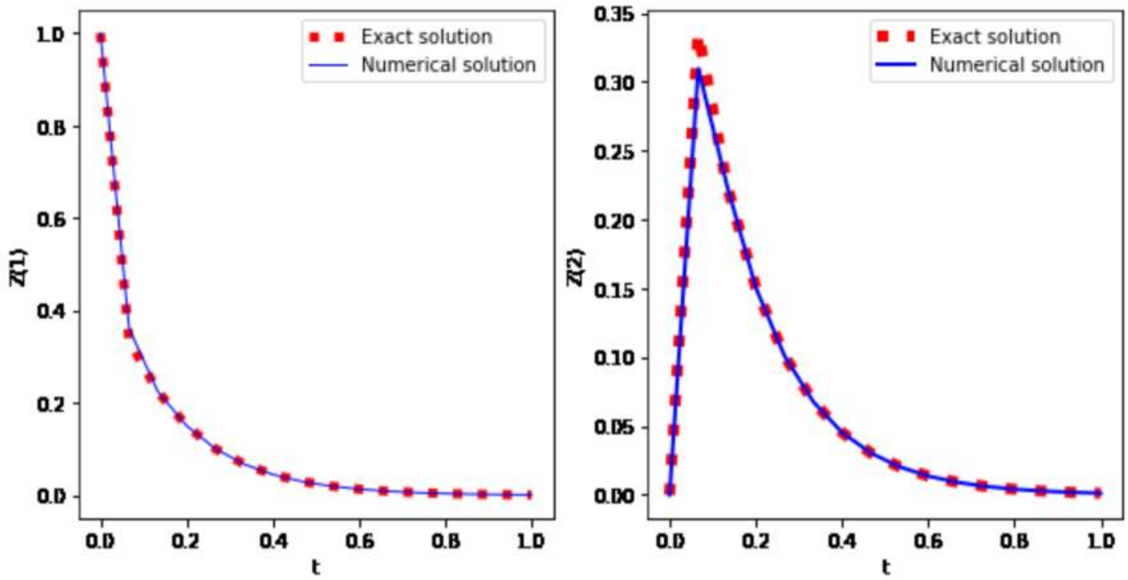


Figure 1: Profile solution for Problem 1

Table 2: Comparison Result of z_1 for Problem 1

t	$z(t)$	$z_1(t)$	$\lvert z(t) - z_1(t)\rvert$
0.20	0.33530156446464362999	0.06767200273471479024 7	2.54284×10^{-2}
0.40	0.22466441197379730427	0.22469925984189926044	3.48478×10^{-5}
0.60	0.15059710594701431165	0.15059886462150065052	1.75860×10^{-6}
0.80	0.10094825899733647829	0.10097264863920704481	2.43896×10^{-5}
1.00	0.06766764161830634611	0.06767200273471479024 7	4.36116×10^{-6}
1.20	0.04535897664470625168	0.04536118382991449059 3	2.20718×10^{-6}
1.40	0.03040503131260898249	0.03040650981639935594 2	1.47850×10^{-7}
1.60	0.02038110198918310758	0.02038209418731861997 1	9.92198×10^{-7}
1.80	0.01366186122364628040	0.01366284629344548635 3	9.85069×10^{-7}
2.00	0.00915781944436709014	0.00915848749567696864 0	6.68051×10^{-7}
2.20	0.00613866995153422058	0.00613911775626054223 58	4.47804×10^{-7}
2.40	0.00411487352451001442	0.00411517371915945535 02	3.00194×10^{-7}
2.60	0.00275828221038038621	0.00275854816120031169 88	2.65950×10^{-7}
2.80	0.00184893185824146541	0.00184911699692458133 3	1.79841×10^{-7}
3.00	0.00123937608833317921	0.00123949663897396519 14	1.20550×10^{-7}

Table 3: Comparison Result of z_2 for Problem 1

t	$z(t)$	$z_2(t)$	$\lvert z(t) - z_2(t)\rvert$
0.20	0.33535037428834497180	0.30960572256641213744	2.57446×10^{-2}
0.40	0.22466451974417424401	0.22464062855752519152	2.38911×10^{-5}
0.60	0.15059710593100098342	0.15060266988864076734	5.56395×10^{-6}
0.80	0.10094825899732591356	0.10092877892490726787	1.94800×10^{-6}

1.00	0.06766764161830634894	0.06766974703465251893	2.10541×10^{-6}
1.20	0.04535897664470625168	0.04536118109942907980	2.20445×10^{-6}
1.40	0.03040503131260898249	0.03040650998570001563	1.47867×10^{-7}
1.60	0.02038110198918310758	0.02038209226510866320	9.90275×10^{-7}
1.80	0.01366186122364628040	0.01366284619401490224	9.84970×10^{-7}
2.00	0.00915781944436709014	0.00915848749555003001	6.68051×10^{-7}
2.20	0.00613866995153422058	0.00613911775626806871	4.47804×10^{-7}
2.40	0.00411487352451001442	0.00411517371907533865	3.00194×10^{-7}
2.60	0.00275828221038038620	0.00275854816119593290	2.65950×10^{-7}
2.80	0.00184893185824146541	0.00184911169969245197	1.79841×10^{-7}
3.00	0.00123937608833317921	0.00123949663897396534	1.20550×10^{-7}

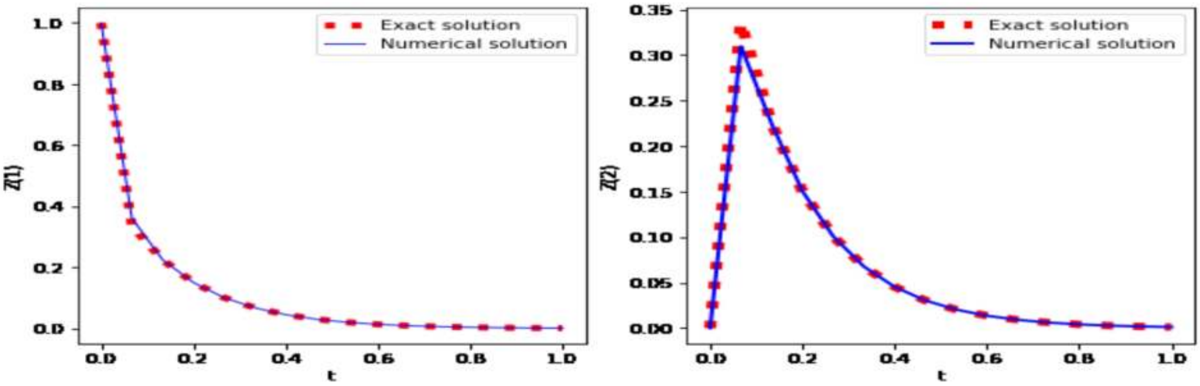


Figure 2: Profile solution for Problem 2

Table 4: Comparison Result of z_1 for Problem 2

t	$z(t)$	$z_1(t)$	$\lvert z(t)-z_1(t)\rvert$
0.0625	1.78123884342673570357 14	1.72247412014931656559 75	5.87647×10^{-2}
0.1250	1.5741655206295851644515	1.5735784574373308188485	5.87063×10^{-7}
0.1875	1.3892017181724113003316	1.3892024037603170774055	6.85587×10^{-7}
0.2500	1.2259662270401725672575	1.2259388994057635266342	2.73276×10^{-7}
0.3125	1.0819113980702038580181	1.0819097671638695472152	1.63090×10^{-9}
0.3750	0.9547834576680082137107	0.9547834503875126024516	7.28049×10^{-9}
0.4375	0.8425934440310276216596	0.8425934516379089236209	7.60688×10^{-9}
0.5000	0.7435861044954685223724	0.7435861104615844738720	5.96611×10^{-9}
0.5625	0.6562124340221962623557	0.6562124427626918203525	8.74049×10^{-9}
0.6250	0.5791054404620863729971	0.5791054482856815168921	7.82359×10^{-9}
0.6875	0.5110587574776790508945	0.5110587643823243132580	6.90464×10^{-9}
0.7500	0.4510077705127836967800	0.4510077766062636108881	6.09347×10^{-9}
0.8125	0.3980129605191156331026	0.3980129676396217252892	7.12050×10^{9}
0.8750	0.3512452048466444049929	0.3512452111740265736755	6.32738×10^{-9}
0.9375	0.3099728053248554045335	0.3099728109087331320470	5.58387×10^{-9}
1.0000	0.2735500405846426751048	0.2735500455125017088059	4.92785×10^{-9}

Table 5: Comparison Result of z_2 for Problem 2

t	$z(t)$	$z_2(t)$	$\|z(t)-z_2(t)\|$
0.0625	-0.016245038257544897841	0.0425196927492879193219	5.87647×10^{-4}
0.1250	-0.016563954486775427961	-0.015976884280185252955	5.70702×10^{-6}
0.1875	-0.014623160590466903241	-0.014623839988320409986	9.75732×10^{-7}
0.2500	-0.012904907614905720049	-0.012877574517332491221	2.73330×10^{-7}
0.3125	-0.011388541032223374104	-0.011386900616537189755	1.64041×10^{-8}
0.3750	-0.010050352185978799434	-0.010050336396356883139	1.57896×10^{-11}
0.4375	-0.008869404674010816489	-0.008869404771661439535	9.76506×10^{-10}
0.5000	-0.007827222152583879181	-0.007827221491546657376	6.61037×10^{-11}
0.5625	-0.006907499305496802761	-0.006907499354057801868	4.85609×10^{-11}
0.6250	-0.006095846741706172347	-0.006095846823639266986	8.19330×10^{-11}
0.6875	-0.005379565868186095272	-0.005379565940867042176	7.26809×10^{-11}
0.7500	-0.004747450215924038913	-0.004747450280046560113	6.41225×10^{-11}
0.8125	-0.004189610110727532980	-0.004189610185679065957	7.49515×10^{-11}
0.8750	-0.003697317945754151631	-0.003697318012358163204	6.66040×10^{-11}
0.9375	-0.003262871634998477942	-0.003262871693776138244	5.87776×10^{-11}
1.0000	-0.002879474111417291316	-0.002879474163289491153	5.18721×10^{-11}

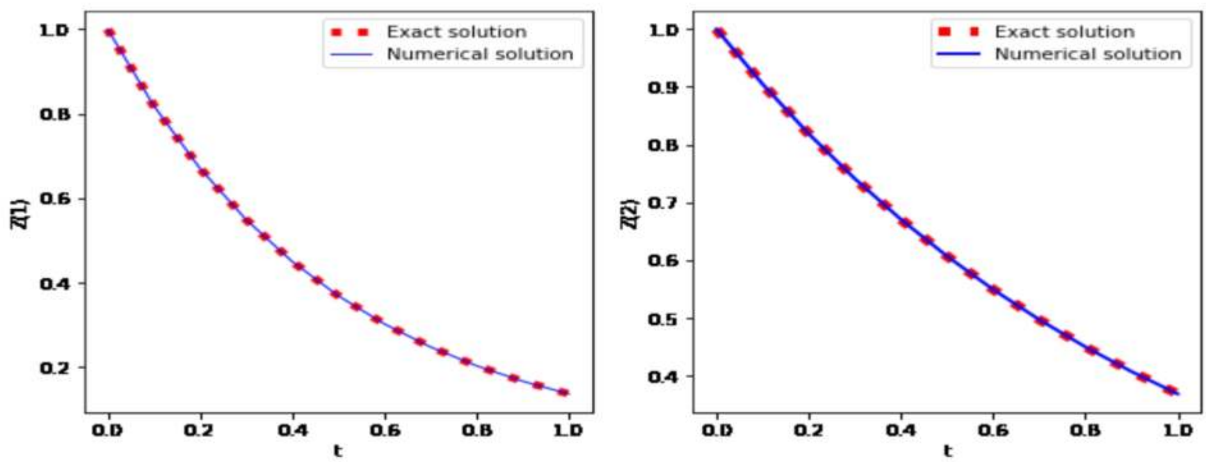


Figure 3: Profile solution for Problem 3

Table 6: Comparison Result of z_1 for Problem 3

t	$Z(t)$	$z_1(t)$	$\|Z(t)-z_1(t)\|$
0.100	0.1353352832366126918	0.135335283900553600380	6.63940×10^{-10}
0.200	0.0183156388887341802	0.018315639037978913619	1.49244×10^{-10}
0.300	0.0024787521766663584	0.002478752208879943511	3.22135×10^{-11}
0.400	0.0003354626279025118	0.000335462633345306306	5.44279×10^{-12}
0.500	0.0000453999297624848	0.000045399930718803095	9.56318×10^{-13}
0.600	$6.144212353328209\times10^{-6}$	$6.14421250257915187\times10^{-6}$	1.49250×10^{-13}
0.700	$8.315287191035678\times10^{-7}$	$8.31528743325846039\times10^{-7}$	2.42222×10^{-14}
0.800	$1.125351747192591\times10^{-7}$	$1.12535178360510108\times10^{-7}$	3.64125×10^{-15}
0.900	$1.522997974471262\times10^{-8}$	$1.52299803111905810\times10^{-8}$	5.66477×10^{-16}
1.000	$2.061153622438557\times10^{-9}$	$2.06115370575375547\times10^{-9}$	8.33151×10^{-17}

Table 7: Comparison Result of z_2 for Problem 3

t	$Z(t)$	$z_2(t)$	$\|Z(t) - z_2(t)\|$
0.100	0.36787944117144232160	0.36787944207382158129	9.02379×10^{-10}
0.200	0.13533528323661269189	0.13533528378795869403	5.51346×10^{-10}
0.300	0.049787068367863942979	0.049787068691376165830	3.23512×10^{-10}
0.400	0.018315638888734180294	0.018315639037311634828	1.48577×10^{-11}
0.500	0.006737946999085467096	0.006737947070050389561	7.09649×10^{-11}
0.600	0.002478752176666358423	0.002478752206771637136	3.01052×10^{-11}
0.700	0.000911881965554516208	0.000911881978835968039	1.32814×10^{-12}
0.800	0.000335462627902511838	0.000335462633329613693	5.42710×10^{-12}
0.900	0.000123409804086679549	0.000123409806381785372	2.29510×10^{-13}
1.000	0.000045399929762484851	0.000045399930680040142	9.17555×10^{-13}

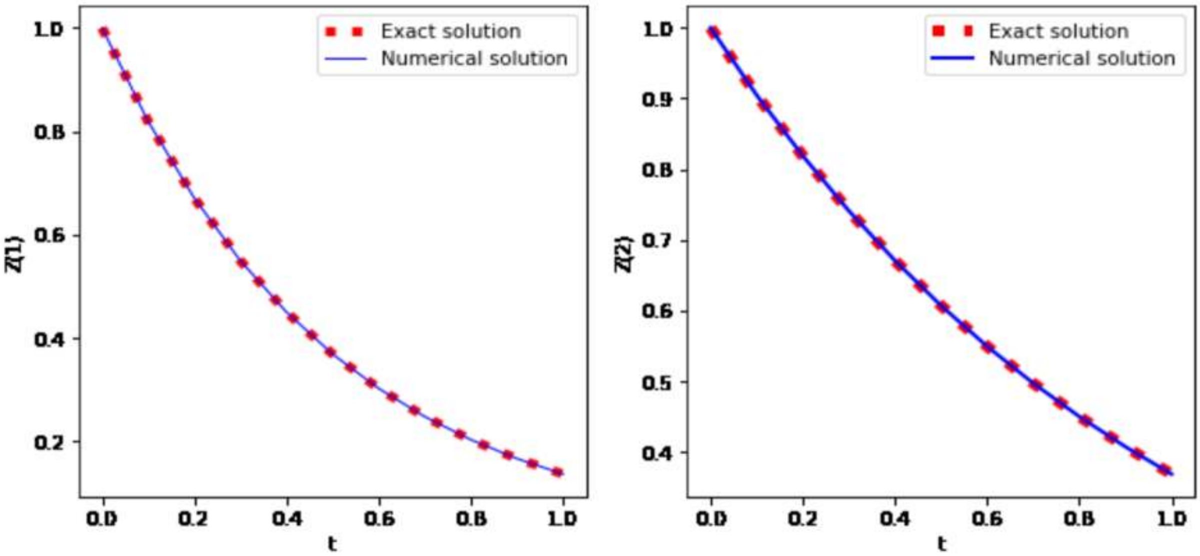


Figure 4: Profile solution for Problem 4

Table 8: Comparison Result of z_1 for Problem 4

t	$Z(t)$	$z_1(t)$	$\|Z(t) - z_1(t)\|$
0.100	0.81873075307798185867	0.81873075717909515030	4.10111×10^{-9}
0.200	0.67032004603563930074	0.67032004732826214578	1.29262×10^{-9}
0.300	0.54881163609402643263	0.54881163715192314629	1.05789×10^{-9}
0.400	0.44932896411722159143	0.44932896498600046686	8.68778×10^{-9}
0.500	0.36787944117144232160	0.36787944368730752857	2.51586×10^{-9}
0.600	0.30119421191220209664	0.30119421304245564822	1.13025×10^{-10}
0.700	0.24659696394160647694	0.24659696486679413398	9.25187×10^{-10}
0.800	0.20189651799465540849	0.20189651875332366913	7.58668×10^{-10}
0.900	0.16529888822158653830	0.16529888964265752646	1.42107×10^{-9}
1.000	0.13533528323661269189	0.13533528398193333375	7.45320×10^{-10}

Table 9: Comparison Result of z_2 for Problem 4

t	$Z(t)$	$z_2(t)$	$\lvert Z(t) - z_2(t)\rvert$
0.100	0.90483741803595957316	0.90483741888208201073	8.46122×10^{-10}
0.200	0.81873075307798185867	0.81873075386712958296	7.89147×10^{-10}
0.300	0.74081822068171786607	0.74081822139576027818	7.14042×10^{-10}
0.400	0.67032004603563930074	0.67032004668177876646	6.46139×10^{-10}
0.500	0.60653065971263342360	0.60653066083442824593	1.12179×10^{-9}
0.600	0.54881163609402643263	0.54881163712358090997	1.02955×10^{-9}
0.700	0.49658530379140951470	0.49658530472298378291	9.31574×10^{-10}
0.800	0.44932896411722159143	0.44932896496017530462	8.42953×10^{-10}
0.900	0.40656965974059911188	0.40656966084988405816	1.10928×10^{-9}
1.000	0.36787944117144232160	0.36787944218432163944	6.63940×10^{-9}

Discussion of Results

The newly derived block Milne technique is applied to stiff initial value problems in ordinary differential equations of the first order. The present technique associates numerical results with their exact solutions and summarizes the results in graphs and tables. The graphs of the exact solutions versus the numerical solutions for problems 1 to 4 are presented in Figures 1 to 4, which demonstrate that the numerical results are in good agreement with the exact solutions. In addition, the absolute errors associated with the numerical results and the analytic solutions are compared in Tables 2–9. The relatively small difference between the exact answer and the computed results proves the validity of the derived technique.

Conclusion

Using the collocation methodology, we established a self-starting hybrid block Milne technique with a greater degree of accuracy in this study. The novel technique aims to increase the efficacy and precision of Linear Multistep techniques by increasing the number of steps at both grid and off-grid locations. In the creation of the new approach, four off-step points and four step points were selected. The convergence of the suggested technique's fundamental attributes was analyzed. Systems of stiff problems were solved numerically to illustrate the accuracy of the suggested technique. The numerical outcomes of the problems demonstrate the effectiveness of the proposed technique, as the computed outcomes corresponded well with the exact solutions. Based on the graphs and tabulated data, we can infer that the proposed technique is an appropriate alternative for dealing with stiff problems that exist in all disciplines of science and engineering. For all computations, Maple 2015 was utilized.

References

Amat, S., Legaz, M. J., & Ruiz-Álvarez, J. (2019). On a variational method for stiff differential equations arising from chemistry kinetics. *Mathematics*, *7*(459), 1-11.

Aslam, M., Farman, M., Ahmad, H., Gia, T. N., Ahmad, A., & Askar, S. (2021). Fractal fractional derivative on chemistry kinetics hires problem. *AIMS Mathematics*, *7*(1), 1155–1184. doi: 10.3934/math2022068.

Badmus, A. M., Yahaya, Y. A. & Pam, Y. C. (2015). Adams Type Hybrid Block Methods Associated with Chebyshev Polynomial for the Solution of Ordinary Differential Equations. *British Journal of Mathematics and Computer Science*, *6*(6), 464-474.

Chapra, S. C., & Canale, R. P. (2015). *Numerical Methods for Engineers*, Seventh Edition. McGraw-Hill Education, New York.

Ehiemua, M. E. & Agbeboh, G. U. (2019). On the Derivationof a new fifth-order Implicit Runge-Kutta Scheme for Stiff Problems in Ordinary Differential Equation. *Journal of Nigerian Mathematical Society*, *38* (2), 247-258.

Eziokwu, C. E. & Okereke, N. R. (2020). On Euler's and Milne's linear multi step methods of s o l v i n g the Ordinary Differential Equations. *Asian Journal of Pure and Applied Mathematics, 2*(2), 1-18.

Garba, J. & Mohammed, U. (2020). Derivation of a New One-Step Numerical Integrator for Solving First Order Ordinary Differential Equations. *Nigerian Journal of Mathematics and Applications,* 30, 155-172.

Gomathi, V. & Rabiyabe, G. (2022). Numerical solution of system of ordinary differential equations using Runge- Kutta method. *International Journal of Mechanical Engineering*. *7*(4), 1444 – 1453.

Iyorter, B. V., Luga, T. & Isah, S. S. (2019). Continuous Implicit Linear Multistep Methods for the Solution of Initial Value Problems of First-Order Ordinary Differential Equations. *IOSR Journal of Mathematics*, *15*(6), 51-64.

Kashkaria, B. S. H. & Syam, M. I. (2019). Optimization of one step block method with three hybrid points for solving first-order ordinary differential equations. *Results in Physics*, 12, 592-596. Doi:10.1016/j.rinp.2018.12.015

Khalsaraei, M. M., Shokri, A., & Molayi, M. (2020). The new class of multistep multiderivative hybrid methods for the numerical solution of chemical stiff systems of first order IVPs. *Journal of Mathematical Chemistry*. https://doi.org/10.1007/s10910-020-01160-z.

Kwanamu, J. A., Skwame, Y. & Sabo, J. (2021). Block hybrid method for solving higher order ordinary differential equation using power series on implicit one-step second derivative. *FUW Trends in Science & Technology Journal*, *6*(2), 576 – 582.

Ma'ali, A. I., Mohammed, U., Audu, K. J., Yusuf, A., & Abubakar, A. D. (2020). Extended Block Hybrid Backward Differentiation Formula for Second Order Fuzzy Differential Equations using Legendre Polynomial as Basis Function. *Journal of Science, Technology, Mathematics and Education*, *16*(1), 100 -111.

Mazarina, M. N., & Syahirbanun, I. (2022). Milne-Simpson method for solving first order fuzzy differential equations using hukuhara approach. *Enhanced Knowledge in Sciences and Technology*, *2*(1), 508-516.

Mohammed, U., Garba, J. & Semenov, M. E. (2021). One-step second derivative block intra-step method for stiff system of ordinary differential equations. Journal of Nigerian Mathematical Society *40*(1), 47-57.

Ndipmong, A.U. & Udechukwu, P. E. (2022). On the analysis of numerical methods for solving first order non-linear ordinary differential equations. *Asian Journal of Pure and Applied Mathematics*, *4*(3), 279-289.

Oghonyon, J. G., Agboola, O. O., Adesanya, A. O., & Ogunniyi, O. P. (2018a). Computing oscillating vibrations employing exponentially fitted block Milne's device. *International Journal of Mechanical Engineering and Technology, 9*(8), 1234–1243.

Oghonyon, J.G., Ehigie, J. & Eke, S. K. (2016a). Investigating the convergence of some selected properties on block predictor-corrector methods and it's applications. *Journal of Engineering and Applied Sciences*, 11, 2402-2408.

Oghonyon, J. G., Okunuga, S. A., Eke, K. S., & Odetunmibi, O. A. *(2018b).* Block Milne's Implementation for Solving Fourth Order Ordinary Differential Equations. *Engineering, Technology and Applied Science Research, 8(3), 2943-2948.* doi: 10.48084/etasr.1914.

Oghonyon, J. G., Okunuga, S. A., & Iyase, S. A. (2016b). Milne's Implementation on block Predictor-corrector methods. *Journal of Applied Sciences, 16*(5), 236-241. doi: 10.3923/jas.2016.236.241.

Tiamiyu, A. T., Cole, A. T., & Audu, K. J. (2021). A Backward Differentiation Formula for Third Order Initial or Boundary Values Problems Using Collocation Method. *Iranian Journal of Optimization*, *13*(2), 81-92.

Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

STATISTICAL ANALYSIS ON DIABETIC PATIENTS: CASE STUDY OF MURTALA MUHAMMED SPECIALIST HOSPITAL KANO, NIGERIA

Abdul Iguda*, Shehu Bala
Department of Mathematical Sciences, Bayero University, Kano.
*CorrespondingAuthor : aiguda.mth@buk.edu.ng

Abstract

Diabetic is a common Disease that is affecting Human life globally. In this research we present the result of Diabetic patients Data obtained from Murtala Muhammed Specialist Hospital for one year which is analyzed using one-way Analysis of Variance (Anova) technique to compare groups of patients across, Sex, Admission, Discharge and Death. From the result we found that there is significant difference between the Means of these four groups of Variables at (0.05) level of significance.

Keywords: Diabetic, *Anova*, Multiple Comparisons and Murtala Muhammed Specialist Hospital

Introduction

Diabetes

Diabetes mellitus (*DM*) is a group of metabolic disorders in which there are high blood sugar levels over a prolonged period. The high blood sugar levels may lead to the symptom which include polyuria, polydipsia, polyphagia. If left untreated diabetes can cause many complications. Acute complication can include diabetic ketoacidosis, hyperosmolar hyperglycemic state, or even death. Serious long-term complications include cardiovascular disease, stroke, chronic kidney disease, food ulcers, and damage to the eyes.

Diabetes is due to either the pancreas not producing enough insulin or the cells of the body not responding properly to the insulin produced There are three main types of diabetes [2].

Clinical Features of Diabetic

Diabetes mellitus is classified into four broad categories; type one, type two, gestational diabetes and ”other specific types”. The other specific types are collection of a few dozen individual causes. Diabetes is more variable disease than once thought and people may have combination of forms. The term ”diabetes ” without qualification, usually refers to diabetes mellitus. Type one *DM* is characterized by loss of the insulin producing beta cells of pancreatic islets, leading to insulin deficiency.

Type two *DM* is characterized by insulin resistance, which may be combined with relatively reduced insulin secretion. The defective responsiveness of body tissues to insulin is believed to involve the insulin receptor. However, the specific defective are not known [2].

The classic symptoms of untreated diabetes are weight loss, polyuria (increase urination), polydipsia (increase thirst), polyphagia (increase hunger). Symptoms may develop rapidly (weeks or months) in type one *DM*, while they usually develop much more slowly and may subtle or absent in type two *DM*. Several other symptoms can mark the onset of diabetes although they are not specific to the disease. In addition to the known ones above, they include blurring vision, headache, fatigue, slow healing of cuts and itching skin. Prolonged high blood glucose can cause glucose absorption in the lens of the eye, which lead to changes in its shape, resulting in vision changes. A number of skin rashes that can occur in diabetes are collectively known as diabetic dermadromes [12].

Causes of Diabetes

In [10], Musman et al. conducted a study ”Pharmaceutical hit of anti-type 2 diabetes mellitus on the phenolic extract of Malaka (Phyllanthus emblica L.) flesh” . The phenolic extract of the phyllanthus emblica was administered to the glucose-induced rats of the Wistar strain Rattus norvegicus for 14 days of the treatment where metformin was used as a positive control. The data generated was analyzed by two-way *Anova* software related to the blood glucose level and by *SAS* software related to histopathological studies at a significant 95% confidence. Their results revealed that the administration of the extract to the rats with a concentration of 100 mg/kg body weight demonstrated a very significant decrease in blood glucose levels and repaired damaged cells better than administering the extract at a concentration of 200 mg/kg body weight. They concluded that the phenolic extract of the Malaka flesh can be utilized as antitype two Diabetes mellitus without damaging other organs.

In [7], Khlaifat et al. conducted a study ”Cross-sectional survey on the diabetes knowledge, risk perceptions and practices among university students in south Jordan” . A self-administered structured questionnaire of N=3000 participants from seven universities campuses were administered about their diabetes knowledge, risk perception and practices in south Jordan. They considered only 2158 (1031 Male and 1127 Females) with ages ranging between 18 to 50 years (97.2% <30 years) were included in the final analysis. Their results shows that 41.9% of the participants have poor diabetes knowledge, 52.5% of the participants have moderate perception on the risk of diabetes, and 61.9% of the participants have slightly higher practice. They concluded that; the university students knowledge, perceived risks and practices towards the disease were not adequate, and they recommended that programs aiming to increase awareness about diabetes for students in all levels and for the general public should be initiated in order to help prevent or delay occurrence of the disease.

In [6], Khadayat et al. performed a study ”Evaluation of the alpha-amylase inhibitory activity of Nepalese medicinal plants used in the treatment of diabetes mellitus”. They used a microtiter plate approach to assess inhibitory activity against alpha-amylase of methanolic extracts of thirty-two medicinal plants. A starch tolerance test was used in rats to investigate the in-vivo study of the methanolic extracts concerning glibenclamide as the positive control. The data obtained was analyzed using one-way *Anova* and further analyzed by Dunnett's one side comparison by SPSS V19. Their result shows that Acacia catechu, Dioscoreabulbifera, and Swertiachirata exhibited inhibitory activity against alpha-amylase and with IC_{50} values; 49.9, 296.1, and 413.5$\mu g/mL$, respectively. They concluded that enzymic assay for alpha-amylase inhibition using extracts was successfully evaluated. Also, the in-vitro and in-vivo study model revealed that medicinal plants could be a potent source of alpha-amylase inhibition. So, they could serve as potential candidates for future development with minimal or no adverse side effects.

In [4] , Feng et al. conducted a study ”Stress adaptation disorders play a role in rat gestational diabetes with oxidative stress and glucose transporter-4 expression”. They assigned the rats to a randomly control group and gestational diabetes mellitus (GDM) group. Data were analyzed using mean, one-way *Anova*, and multiple linear regression analysis using SPSS 19.0. They found that stress adaptation existed in GDM rats, and insulin resistance is an important biological basis for the occurrence of GDM. They concluded that attention to stress and stress related hormones besides stricter diet control should be given to prevent adverse perinatal outcomes for patients with GDM.

In [1], Bala et al. performed a study ”Systematic review of the effect of sleep apnea syndrome and its therapy on HbA1c in type 2 diabetes”. They obtained their data through a systematic review of literatures from Embase, PubMed Web of science for studies published from database inception until January 24, 2019. Cross-sectional studies reported no statistically significant difference in HbA1c between those with and without SAS and no linear relation between AH1 and HbA1c. Their findings suggest an effect of hypoxemia during apnea/hypopnea episodes on glycemic control and do not support any effect of CPAP on glycemic control in type two diabetes.

Study Area and Data

Murtala Muhammad Specialist Hospital Kano is the biggest tertiary health care institution owned by the state government. Besides its primary function of providing health-care services to the state, it also serves as training and research center for the state higher institution, such as, school of nursing and midwifery, school of hygiene, school of health technology, and so on. Patronage of the hospital is very high due to affordable health care service and availability of all medical sub-specialties' as well as qualified personnel who are well experienced in various fields of specialization. The secondary data for one year from June 2016 to June 2017 was collected for this research work from the health record department of the hospital.

Analysis and Discussion of Result

Table 1: Definition of Variables Used in the Analysis

Abbreviation	Meaning	Abbreviation	Meaning
MD	Male Diabetes Patients	FD	Female Diabetes Patients
ADD	Admitted Diabetes Patients	DISD	Discharged Diabetes Patients
DD	Dead Diabetes Patient		

Analysis of variance was used to analyze the secondary data obtained from Murtala Muhammed Specialist Hospital Kano. Analysis of variance is essentially an arithmetic process for partitioning a total sum of squares into components associated with recognized sources of variation.

Table 2: Descriptive

RESPONSE									
	N	Mean	Std. Deviation	Std. Error	95% Confidence Interval for Mean		Minimum Maximum	
					Lower Bound	Upper Bound		
MD	12	357.9167	119.95337	34.62755	281.7019	434.1314	220.00	581.00
FD	12	484.4167	240.00396	69.28318	331.9254	636.9079	30.00	950.00
ADD	12	842.3333	239.26681	69.07038	690.3105	994.3562	515.00	1490.00
DISD	12	787.2500	228.66853	66.01092	641.9609	932.5391	497.00	1389.00
DD	12	55.0833	35.81127	10.33782	32.3299	77.8367	16.00	112.00
Total	60	505.4000	345.09160	44.55113	416.2534	594.5466	16.00	1490.00

Descriptives table gives us information on the Mean, Std. Deviation, Std. Error and the number of cases for each group

Table 3: Test of Homogeneity of Variances

RESPONSE .			
Levene Statistic	df_1	df_2	Sig.
2.599	4	55	0.046

Test of homogeneity of variances table. If the significance level of the Levene statistic that is P_{value} is greater than or equal to 0.05, then *Anova* is used otherwise Robust Tests of Equality of Means would be used instead of the *Anova*.

Table 4: *Anova* Table for Completely Randomized Design (RCD)

RESPONSE					
Levene Statistic	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	5015282.733	4	1253820.683	34.293	0.000
Within Groups	2010921.667	55	36562.212		
Total	7026204.400	59			

From the *Anova* table if the significance P_{value} is less than 0.05, then there is significance difference in the Means somewhere across the groups between the four variables. But *Anova* does not tell us which of the Means are really difference until we go to multiple comparisons. If *Anova* is used, then Turkey *HSD* will be used for multiple comparisons

Table 5: Robust Tests of Equality of Me ans

RESPONSE				
	Statistic[a]	df_1	df_2	Sig.
Brown Forsythe	34.293	4	38.542	0.000
a. Asymptotically F distributed				

If the significance P_{value} of the Robust Test of Equality of Means is less than 0.05, then there is significance difference somewhere across the Means of four groups of variales.

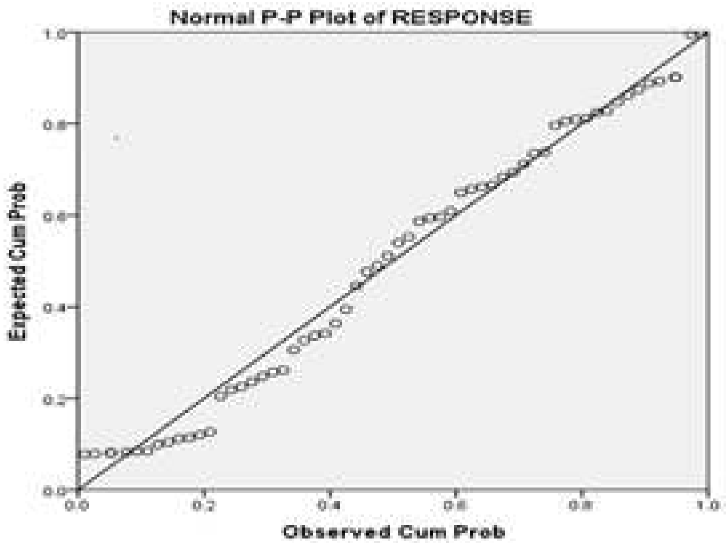
Abacus (Mathematics Science Series)
Vol. 49, No 4, December, 2022

Table 6: Multiple Comparisons

Dependent Variable: RESP							
	Factors		Mean Difference (I-J)	Std. Error	Sig.	95% Confidence Interval	
	I	J				Lower Bound	Upper Bound
Turkey HSD	MD	FD	- 126.50000	78.06217	0.491	- 346.6609	93.6609
		ADD	- 484.41667*	78.06217	0.000	- 704.5775	- 264.2558
		DISD	- 429.33333*	78.06217	0.000	- 649.4942	- 209.1725
		DD	302.83333*	78.06217	0.003	82.6725	522.9942
	FD	MD	126.50000	78.06217	0.491	- 93.6609	346.6609
		ADD	- 355.66667*	78.06217	0.000	- 578.0775	- 137.7558
		DISD	- 302.83333*	78.06217	0.003	- 522.9942	- 82.6725
		DD	429.33333*	78.06217	0.000	209.1725	649.4942
	ADD	MD	484.41667*	78.06217	0.000	264.2558	704.5775
		FD	357.91667*	78.06217	0.000	137.7558	578.0775
		DISD	55.08333	78.06217	0.954	- 165.0775	275.2442
		DD	787.25000*	78.06217	0.000	567.0891	1007.4109
	DISD	MD	429.33333*	78.06217	0.000	209.1725	649.4942
		FD	302.83333*	78.06217	0.003	82.6725	522.9942
		ADD	- 55.08333	78.06217	0.954	- 275.2442	165.0775
		DD	732.16667*	78.06217	0.000	512.0058	952.3275
	DD	MD	- 302.83333*	78.06217	0.003	- 522.9942	- 82.6725
		FD	- 429.33333*	78.06217	0.000	- 649.4942	- 209.1725
		ADD	- 787.25000*	78.06217	0.000	- 1007.4109	- 567.0891
		DISD	- 732.16667*	78.06217	0.000	- 952.3275	- 512.0058

Games-Howell	MD	FD	- 126.50000	77.45467	0.499	- 363.4965	110.4965
		ADD	- 484.41667*	77.26438	0.000	- 720.7830	- 248.0503
		DISD	- 429.33333*	74.54199	0.000	- 656.6900	- 201.9767
		DD	302.83333*	36.13777	0.000	188.9767	416.6900
	FD	MD	126.50000	77.45467	0.499	- 110.4965	363.4965
		ADD	- 357.91667*	97.83085	0.011	- 648.1790	- 67.6543
		DISD	- 302.83333*	95.69535	0.033	- 586.8159	- 18.8507
		DD	429.33333*	70.05019	0.000	204.4658	654.2008
	ADD	MD	484.41667*	77.26438	0.000	248.0503	720.7830
		FD	357.91667*	97.83085	0.011	67.6543	648.1790
		DISD	55.08333	95.54140	0.977	- 228.4355	338.6021
		DD	787.25000*	69.83973	0.000	563.0679	1011.4321
	Factors		Mean Difference (I-J)	Std. Error	Sig.	95% Confidence Interval	
	I	J				Lower Bound	Upper Bound
Games-Howell	DISD	MD	429.33333*	74.54199	0.000	201.9767	656.6900
		FD	302.83333*	95.69535	0.033	18.8507	586.8159
		ADD	- 55.08333	95.54140	0.977	- 338.6021	228.4355
		DD	732.16667*	66.81551	0.000	517.8367	946.4967
	DD	MD	- 302.83333*	36.13777	0.000	- 416.6900	- 188.9767
		FD	- 429.33333*	70.05019	0.000	- 654.2008	- 204.4658
		ADD	- 787.25000*	69.83973	0.000	- 1011.4321	- 563.0679
		DISD	- 732.16667*	66.81551	0.000	- 946.4967	- 517.8367
*: The mean difference is significant at the 0.05 level.							

From our multiple comparisons table under Turkey HSD since *Anova* is used any value with a steric (*) means there is significant different between the Means of these four groups of variables. From the graph below the normal



probability plot indicates that our data is normally distributed, which agrees with one of the assumptions of *Anova*.

Conclusion

In this research work, we found that females are more susceptible to diabetes than their male counter part on the the average. We have also found that number of discharges are significantly higher than the number of deaths on the average, this indicates that on the average, the number of people who manage diabetes is much higher than the number of those who die as a result of the disease.

References

C. Bala, G. Roman, D. Ciobanu and A. Rasu, *A systematic review of the effect of sleep apnea syndrome and its therapy on HbA1c in type 2 diabetes,* Springer - International Journal of Diabetes in Developing Countries, https://doi.org./10.1007/s13410-019-00784-5,(2020).

G. David and D. Gardner, *Greenspan's Basic and Clinical Endocrinilogy*, New York McGraw-Hill Medical, 9th Edition (2011).

W. G. Cochran and G. M. Cox, *Experimental Designs*, Second Edition, John Wiley and Sons,(1957).

Y. Feng, Q. Feng, S. Yin, X. Xu, X. Song, H. Qu and J. Hu, *Stress adaptation disorders play a role in rats gestational diabetes with oxidative stress and glucose transporter-4 expression*, Taylor and Francis Gynecological Endocrinology, https://doi.org/10.1080/09513590.2019.1707797,(2020), 1-6.

P. W. M. John, *Statistical Design and Analysis of Experiments,* Macrnillan, New York.(1971).

K. Khadayat, B. P. Marasini, H. Gautam, S. Ghaju and N. Parajuli, *Evaluation of the alpha-amylase inhibitory activity of Nepalese medicinal plants used in the treatment of diabetes mellitus*, Springer -Clinical Phytoscience, https://doi.org/10.1186/s40816-020-00179-8,6:34(2020).

A. M. Khlaifat, L. A. Al-hadid, R. S. Dabbour and N. Shoqirat, *Cross-sectional survey on the diabetes knowledge, risk perceptions and practices among university students in South Jordan*, Springer -Journal of Diabetes and Metabolic Disorders, https://doi.org/10.1007/s40200-020-00571-8, (2020).

A. E. Kitabchi et al., *Hyperglycemic crises in adult patients with Diabetes*, Diabetic Care 32:7(2009).

C. Y. Kramer and S. Glass, *Analysis of variance of a Latin square design with missing observations*. Appl. Stat., 9(I960), 43-47.

M. Musman, M. Zakia, F. F. I. Rahmayani, E. Erlidawati and S. Safrida, *Pharmaceutical hit of anti type 2 Diabetes mellitus on the phenolic extract of Malaka (Phyllanthus emblica L.) flesh*. Springer - Clinical Phytoscience, https://doi.org/10.1186/s40816/-019-0138-7, 5:47(2019), 1-10.

World Health Organization, *About Diabetes* Archived from original on 31 march 2014. Retrieved 4 April 2014.

J. D. Rockefeller, Diabetes, Symtoms, Causes, Treatment and Prevention, (2015).

Denise Syndercombe-*Diabetes care*, Medical sciences , 32 (7)(2014),1335-43.

L . Welch -*On the z-test in randomized blocks and Latin squares*, Biometrika London , (1937),29-31.

M. B. Wilk-*The randomization analysis of a generalized randomized block Design*,Biometrika, 42(1955),70-79.

E. J. Williams -*Experimental designs balanced for the estimation of residual effects of treatments*, Australian J. Sci. Research, A, 2, (1949), 149-151.

REMARKS ON TRIPLED FIXED POINT STABILITY FOR ITERATIVE PROCEDURES IN CONTRACTIVE TYPE MAPPINGS

Aniki, S. A.
Department of Mathematics and Statistics, Faculty of Science,
Confluence University of Science and Technology, Osara, Kogi State.
anikisa@custech.edu.ng

Abstract

Fixed point theorems have become the focus of interest recently, specifically for their potential applications in iterative procedures. This work expatiate the notion of stability of tripled fixed point iterative procedures and establish results for mixed monotone mappings which satisfy contractive-type conditions. The findings complement existing results in the literature.

Keywords: Tripled fixed point, stability, contractive condition, mixed monotone operator

Introduction

Metrical fixed point theory has significantly improved on the approaches of mathematics through the Banach contraction concept to sciences and its applications. This concept is a classical and powerful tool in nonlinear analysis because of its very useful structure.

The Banach concept was applied on partially ordered complete metric spaces and starting from the results, Bhaskar and Laksmikantham (2006) extend this theory to partially ordered metric spaces and introduce the concept of coupled fixed point for mixed-monotone operators of Picard type, obtaining results involving the existence and uniqueness of the coincidence points for mixed monotone operators $T: X^2 \rightarrow X$ in the presence of a contractive condition. This concept of coupled fixed points in partially ordered metric and cone metric spaces have been studied by several authors, including Ciric and Lakshmikantham (2009), Lakshmikantham and Ciric (2009), and Sabetghadam, Masiha and Sanatpour (2009), Karapinar (2010), Choudhury and Kundu (2010), Aniki and Rauf (2019).

Recently, Berinde and Borcut (2011) obtained extensions to the concept of tripled fixed points and tripled coincidence fixed points and also obtained tripled fixed points theorems and tripled coincidence theorems for mappings in partially ordered metric spaces. Work on tripled fixed point was advanced by Abbas, Aydi and Karapinar (2011), Amini-Harandi (2012) and Kishore (2011).

Very recently, Rauf and Aniki (2020) introduced quadrupled fixed point theorems for contractive type mappings in partially ordered cauchy spaces. Also, following the series, Aniki and Rauf (2021) established the stability theorem and results for quadrupled fixed point of contractive type single valued operators. On the other hand, by adapting the stability concept of the iterative fixed point method, Olatinwo (2012) tested the stability of the related iterative fixed point method using several contractive conditions for which the existence of a unique coupled fixed point has been demonstrated in the literature.

Methodology

The following basic notations are useful in the statement and in proving of our Main result.

Definition 1. (Berinde & Borcut, 2011). Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Then, the product space X^3 has the following partial order

$$(p, q, r) \leq (s, t, u) \Leftrightarrow s \geq p, t \leq q, u \geq r; \quad (p, q, r), (s, t, u) \in X^3$$

Definition 2. (Berinde & Borcut, 2011). Let $(X, \leq)$ be a partially ordered set and $T: X^3 \to X$ be a mapping. We say that T has a mixed monotone property if $T(s, t, u)$ is monotone nondecreasing in s, monotone nonincreasing in t, and monotone nondecreasing in u, that is for any $s, t, u \in X$,

$$s_1 \leq s_2 \Rightarrow T(s_1, t, u) \leq T(s_2, t, u), \quad s_1, s_2 \in X,$$
$$t_1 \leq t_2 \Rightarrow T(s, t_1, u) \geq T(s, t_2, u), \quad t_1, t_2 \in X,$$
$$u_1 \leq u_2 \Rightarrow T(s, t, u_1) \leq T(s, t, u_2), \quad u_1, u_2 \in X.$$

Definition 3. (Timis, 2014). An element $(s, t, u) \in X^3$ is called tripled fixed point of the mapping $T: X^3 \to X$, if

$$T(s, t, u) = s, \qquad T(t, s, t) = t, \qquad T(u, t, s) = u.$$

Definition 4. (Timis, 2014). A mapping $T: X^3 \to X$ is said to be (κ, μ, ψ)-contraction if and only if there exist three constants $\kappa \geq 0, \mu \geq 0, \psi \geq 0, \kappa + \mu + \psi < 1$, such that $\forall\, s, t, u, p, q, r \in X$,

$$d\big(T(s, t, u), T(p, q, r)\big) \leq \kappa d(s, p) + \mu d(t, q) + \psi d(u, r) \tag{1}$$

From (1) above, we introduce some new contractive conditions

Let (X, d) be a metric space. For a map $T: X^3 \to X$ there exists $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \geq 0$, with $\alpha_1 + \alpha_2 + \alpha_3 < 1$, $\beta_1 + \beta_2 + \beta_3 < 1$, such that $\forall\, s, t, u, p, q, r \in X$. Now, we introduce the following definitions of contractive conditions:

i.
$$d\big(T(s, t, u), T(p, q, r)\big) \leq \alpha_1 d(T(s, t, u), s) + \beta_1 d(T(p, q, r), p); \tag{2}$$
$$d\big(T(t, s, t), T(q, p, q)\big) \leq \alpha_2 d(T(t, s, t), t) + \beta_2 d(T(q, p, q), q); \tag{3}$$
$$d\big(T(u, t, s), T(r, q, p)\big) \leq \alpha_3 d(T(u, t, s), u) + \beta_3 d(T(r, q, p), r); \tag{4}$$

ii.
$$d\big(T(s, t, u), T(p, q, r)\big) \leq \alpha_1 d(T(s, t, u), p) + \beta_1 d(T(p, q, r), s); \tag{5}$$
$$d\big(T(t, s, t), T(q, p, q)\big) \leq \alpha_2 d(T(t, s, t), q) + \beta_2 d(T(q, p, q), t); \tag{6}$$
$$d\big(T(u, t, s), T(r, q, p)\big) \leq \alpha_3 d(T(u, t, s), r) + \beta_3 d(T(r, q, p), u). \tag{7}$$

Let $A, B \in M_{(m,n)}(\mathbb{R})$ be two matrices. We write $A \leq B$, if $a_{ij} \leq b_{ij}$ for all $i = \overline{1, m}$, $j = \overline{1, n}$.

Lemma 1. (Timis, 2014). Let $\{a_n\}, \{b_n\}$ be sequences of nonnegative numbers and h be a constant, such that $0 \leq h < 1$ and $a_{n+1} \leq h a_n + b_n, n \geq 0$,

If $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

We also give the following result which extends Lemma 1 to linear sequences.

Lemma 2. (Timis, 2014). Let $\{p_n\}, \{q_n\}, \{r_n\}$ be sequences of nonnegative real numbers, consider a matrix $A \in M_{(3,3)}(\mathbb{R})$ with nonnegative elements, such that

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \\ r_{n+1} \end{pmatrix} \leq A \cdot \begin{pmatrix} p_n \\ q_n \\ r_n \end{pmatrix} + \begin{pmatrix} \varepsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix}, \quad n \geq 0, \tag{8}$$

with

i. $\lim\limits_{n\to\infty} A^n = 0_3$;

ii. $\sum_{k=0}^{\infty} \varepsilon_k < \infty$, $\sum_{k=0}^{\infty} \delta_n < \infty$, $\sum_{k=0}^{\infty} \gamma_n < \infty$.

$\lim_{n\to\infty} \begin{pmatrix} \varepsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\lim_{n\to\infty} \begin{pmatrix} p_n \\ q_n \\ r_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

1. Main Results

Let (X, d) be a metric space and $T: X^3 \to X$ a mapping. For $(s_0, t_0, u_0) \in X^3$ the sequence $\{(s_n, t_n, u_n)\} \subset X^3$ defined by

$$s_{n+1} = T(s_n, t_n, u_n), \ \ t_{n+1} = T(t_n, s_n, t_n), \ \ u_{n+1} = T(u_n, t_n, s_n) \tag{9}$$

for $n = 0,1,2, \ldots$, is said to be tripled fixed point iterative procedures.

We give the following stability definition with respect to T, in metric spaces, relative to tripled fixed points iterative procedures.

Definition 5. Let (X, d) be a complete metric space and

$$Fix_t(T) = \{(s^*, t^*, u^*) \in X^3 | T(s^*, t^*, u^*) = s^*, T(t^*, s^*, t^*) = t^*, T(u^*, t^*, s^*) = u^*\}$$

is the set of tripled fixed points of T.

Let $\{(s_n, t_n, u_n)\} \subset X^3$ be the sequence generated by the iterative procedure defined by (9), where $(s_0, t_0, u_0) \in X^3$ is the initial value, which converges to a tripled fixed point (s^*, t^*, u^*) of T.

Let $(p_n, q_n, r_n) \subset X^3$ be an arbitrary sequence. For all $n = 0,1,2, \ldots$

we set

$$\varepsilon_n = d(p_{n+1}, T(p_n, q_n, r_n)), \delta_n = d\big(q_{n+1}, T(q_n, r_n, q_n)\big), \gamma_n = d\big(r_{n+1}, T(r_n, q_n, p_n)\big).$$

Then, the tripled fixed point iterative procedure defined by (9) is $T-$stable or stable with respect to T, if and only if

$$\lim_{n\to\infty} (\varepsilon_n, \delta_n, \gamma_n) = 0_{\mathbb{R}^3}$$
$$\Longrightarrow \lim_{n\to\infty} (p_n, q_n, r_n) = (s^*, t^*, u^*).$$

Theorem 1. Let $(X, \leq)$ be a partially ordered set. Suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T: X^3 \to X$ be a continuous mapping having a mixed monotone property on $X$and satisfying the contraction condition (1).

If there exists $s_0, t_0, u_0 \in X$ such that

$$s_0 \leq T(s_0, t_0, u_0), t_0 \geq T(t_0, s_0, t_0) \ and \ u_0 \leq T(u_0, t_0, s_0)$$

then, there exist $s^*, t^*, u^* \in X$ such that

$$s^* = T(s^*, t^*, u^*), t^* = T(t^*, s^*, t^*) \text{ and } u^* = T(u^*, t^*, s^*)$$

Assume that for every $(s, t, u), (s_1, t_1, u_1) \in X^3$, then there exists $(p, q, r), (p_1, q_1, r_1) \in X^3$ that is comparable to (s, t, u) and (s_1, t_1, u_1). For $(s_0, t_0, u_0) \in X^3$, let $\{(s_n, t_n, u_n)\} \subset X^3$ be the tripled fixed point iterative procedure defined by (9). Then, the tripled fixed point iterative procedure is $T-$stable.

Corollary 1. Let $(X, \leq)$ be a partially ordered set. Suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T: X^3 \to X$ be a continuous mapping having a mixed monotone property on X.

There exists $h \in [0,1)$, such that T satisfies the following contraction condition.

$$d\big(T(s, t, u), T(p, q, r)\big) \leq \frac{h}{3}[d(s, p) + d(t, q) + d(u, r)], \tag{10}$$

for each $s,t,u,p,q,r \in X$, with $s \geq p, t \leq q$ and $u \geq r$
If there exists $s_0, t_0, u_0 \in X$ such that

$$s_0 \leq T(s_0,t_0,u_0), t_0 \geq T(t_0,s_0,t_0), \text{ and } u_0 \leq T(u_0,t_0,s_0)$$

Then, there exist $s^*, t^*, u^* \in X$ such that

$$s^* = T(s^*,t^*,u^*),\ t^* = T(t^*,s^*,t^*) \text{ and } u^* = T(u^*,t^*,s^*).$$

Assume that for every $(s,t,u),(s_1,t_1,u_1) \in X^3$, then there exists $(p,q,r),(p_1,q_1,r_1) \in X^3$ that is comparable to (s,t,u) and (s_1,t_1,u_1). For $(s_0,t_0,u_0) \in X^3$, let $\{(s_n,t_n,u_n)\} \subset X^3$ be the tripled fixed point iterative procedure defined by (9). Then, the tripled fixed point procedure is T −stable.

Proof.

On applying theorem 1, for $\kappa = \mu = \psi := \frac{h}{3}$.
Let $(p_n,q_n,r_n) \subset X^3$ be an arbitrary sequence. For all $n = 0,1,2,\ldots$
then

$$\varepsilon_n = d(p_{n+1},T(p_n,q_n,r_n)), \delta_n = d\big(q_{n+1},T(q_n,r_n,q_n)\big), \gamma_n = d\big(r_{n+1},T(r_n,q_n,p_n)\big),$$

Taking $\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \gamma_n = 0,$
To be able to establish that$\lim_{n\to\infty} p_n = s^*, \lim_{n\to\infty} q_n = t^*$and $\lim_{n\to\infty} r_n = u^*$.
Therefore, using contraction condition (10), we have

$$\begin{aligned} d(p_{n+1},s^*) &\leq d\big(p_{n+1},T(p_n,q_n,r_n)\big) + d(T(p_n,q_n,r_n),s^*) \\ &= d\big(T(p_n,q_n,r_n),T(s^*,t^*,u^*)\big) + \varepsilon_n \\ &\leq \frac{h}{3}[d(p_n,s^*) + d(q_n,t^*) + d(r_n,u^*)] + \varepsilon_n, \end{aligned} \tag{11}$$

$$\begin{aligned} d(q_{n+1},t^*) &\leq d\big(q_{n+1},T(q_n,p_n,q_n)\big) + d(T(q_n,p_n,q_n),t^*) \\ &= d\big(T(q_n,p_n,q_n),T(t^*,s^*,t^*)\big) + \delta_n \\ &\leq \frac{h}{3}[d(q_n,t^*) + d(p_n,s^*) + d(q_n,t^*)] + \delta_n, \\ &= \frac{h}{3}[d(p_n,s^*) + 2d(q_n,t^*)] + \delta_n, \end{aligned} \tag{12}$$

$$\begin{aligned} d(r_{n+1},u^*) &\leq d\big(r_{n+1},T(r_n,q_n,p_n)\big) + d(T(r_n,q_n,p_n),u^*) \\ &= d\big(T(r_n,q_n,p_n),T(u^*,t^*,s^*)\big) + \gamma_n \\ &\leq \frac{h}{3}[d(r_n,u^*) + d(q_n,t^*) + d(p_n,s^*)] + \gamma_n. \end{aligned} \tag{13}$$

Now, from (11), (12) and (13), we obtain

$$\begin{pmatrix} d(p_{n+1},s^{\cdot}) \\ d(q_{n+1},t^{\cdot}) \\ d(r_{n+1},u^{\cdot}) \end{pmatrix} \leq \begin{pmatrix} \frac{h}{3} & \frac{h}{3} & \frac{h}{3} \\ \frac{h}{3} & \frac{2}{3}h & 0 \\ \frac{h}{3} & \frac{h}{3} & \frac{h}{3} \end{pmatrix} \cdot \begin{pmatrix} d(p_n,s^{\cdot}) \\ d(q_n,t^{\cdot}) \\ d(r_n,u^{\cdot}) \end{pmatrix} + \begin{pmatrix} \varepsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix},$$

where,

$$0 \leq \frac{h}{3} + \frac{h}{3} + \frac{h}{3} = h < 1,$$

Applying Lemma 2, we need $A^n \to 0$, as $n \to \infty$.
By a way of simplification, let

$$A \coloneqq \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix}$$

where

$$a_1 + b_1 + c_1 = d_1 + e_1 + f_1 = g_1 + h_1 + i_1 = \frac{h}{3} + \frac{h}{3} + \frac{h}{3} = h < 1 \tag{14}$$

Then,

$$A^2 = \begin{pmatrix} \frac{h}{3} & \frac{h}{3} & \frac{h}{3} \\ \frac{h}{3} & \frac{2}{3}h & 0 \\ \frac{h}{3} & \frac{h}{3} & \frac{h}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{h}{3} & \frac{h}{3} & \frac{h}{3} \\ \frac{h}{3} & \frac{2}{3}h & 0 \\ \frac{h}{3} & \frac{h}{3} & \frac{h}{3} \end{pmatrix},$$

$$= \begin{pmatrix} \frac{h^2}{9} + \frac{h^2}{9} + \frac{h^2}{9} & \frac{h^2}{9} + \frac{2}{9}h^2 + \frac{h^2}{9} & \frac{h^2}{9} + \frac{h^2}{9} \\ \frac{h^2}{9} + \frac{2}{9}h^2 & \frac{h^2}{9} + \frac{4}{9}h^2 & \frac{h^2}{9} \\ \frac{h^2}{9} + \frac{h^2}{9} + \frac{h^2}{9} & \frac{h^2}{9} + \frac{2}{9}h^2 + \frac{h^2}{9} & \frac{h^2}{9} + \frac{h^2}{9} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{h^2}{3} & \frac{4}{9}h^2 & \frac{2}{9}h^2 \\ \frac{h^2}{3} & \frac{5}{9}h^2 & \frac{h^2}{9} \\ \frac{h^2}{3} & \frac{4}{9}h^2 & \frac{2}{9}h^2 \end{pmatrix} \coloneqq \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{pmatrix}$$

where

$$a_2 + b_2 + c_2 = d_2 + e_2 + f_2 = g_2 + h_2 + i_2 = h^2 < h < 1. \tag{15}$$

Then,

$$A^3 = A^2 \cdot A$$

$$= \begin{pmatrix} \frac{h^2}{3} & \frac{4}{9}h^2 & \frac{2}{9}h^2 \\ \frac{h^2}{3} & \frac{5}{9}h^2 & \frac{h^2}{9} \\ \frac{h^2}{3} & \frac{4}{9}h^2 & \frac{2}{9}h^2 \end{pmatrix} \cdot \begin{pmatrix} \frac{h}{3} & \frac{h}{3} & \frac{h}{3} \\ \frac{h}{3} & \frac{2}{3}h & 0 \\ \frac{h}{3} & \frac{h}{3} & \frac{h}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{h^3}{9} + \frac{4}{27}h^3 + \frac{2}{27}h^3 & \frac{h^3}{9} + \frac{8}{27}h^3 + \frac{2}{27}h^3 & \frac{h^3}{9} + \frac{2}{27}h^3 \\ \frac{h^3}{9} + \frac{5}{27}h^3 + \frac{h^3}{27} & \frac{h^3}{9} + \frac{10}{27}h^3 + \frac{h^3}{27} & \frac{h^3}{9} + \frac{2}{27}h^3 \\ \frac{h^3}{9} + \frac{4}{27}h^3 + \frac{2}{27}h^3 & \frac{h^3}{9} + \frac{8}{27}h^3 + \frac{2}{27}h^3 & \frac{h^3}{9} + \frac{2}{27}h^3 \end{pmatrix}$$

$$\coloneqq \begin{pmatrix} a_3 & b_3 & c_3 \\ d_3 & e_3 & f_3 \\ g_3 & h_3 & i_3 \end{pmatrix}$$

$$a_3 + b_3 + c_3 = d_3 + e_3 + f_3 = g_3 + h_3 + i_3 = h^3 < h^2 < h < 1. \tag{16}$$

Now, we prove by induction that

$$A^n = \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & h_n & i_n \end{pmatrix}$$

where,

$$a_n + b_n + c_n = d_n + e_n + f_n = g_n + h_n + i_n = h^n < h^{n-1} <\cdots< h^2 < h < 1. \tag{17}$$

Taking (17) to be true for n, then

$$A^{n+1} = \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & h_n & i_n \end{pmatrix} \cdot \begin{pmatrix} \frac{h}{3} & \frac{h}{3} & \frac{h}{3} \\ \frac{h}{3} & \frac{2}{3}h & 0 \\ \frac{h}{3} & \frac{h}{3} & \frac{h}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{h}{3}(a_n + b_n + c_n) & \frac{h}{3}(a_n + 2b_n + c_n) & \frac{h}{3}(a_n + c_n) \\ \frac{h}{3}(d_n + e_n + f_n) & \frac{h}{3}(d_n + 2e_n + f_n) & \frac{h}{3}(d_n + f_n) \\ \frac{h}{3}(g_n + h_n + i_n) & \frac{h}{3}(g_n + 2h_n + i_n) & \frac{h}{3}(g_n + i_n) \end{pmatrix}$$

We have

$$a_{n+1} + b_{n+1} + c_{n+1} = \frac{h}{3}(a_n + b_n + c_n) + \frac{h}{3}(a_n + 2b_n + c_n) + \frac{h}{3}(a_n + c_n)$$
$$= h(a_n + b_n + c_n)$$

From (17), we have

$$= h(h^n) = h^{n+1} < h^n < \cdots < h < 1,$$

Similarly,

$$d_{n+1} + e_{n+1} + f_{n+1} = g_{n+1} + h_{n+1} + i_{n+1} = h(h^n) = h^{n+1} < h^n < \cdots < h < 1.$$

Therefore,

$$\lim_{n\to\infty} A^n = 0_3,$$

Now, having satisfied the condition of the hypothesis of Lemma 2, we apply to get

$$\lim_{n\to\infty} \begin{pmatrix} p_n \\ q_n \\ r_n \end{pmatrix} = \begin{pmatrix} s^* \\ t^* \\ u^* \end{pmatrix},$$

So, the tripled fixed point iteration procedure defined by (9) is T −stable.

Conclusion

These findings complete the notion of stability of tripled fixed point iterative procedures and establish results for mixed monotone mappings which satisfy contractive-type conditions, and were able to show that tripled fixed point iteration procedure of this type is T −stable.

References

Amini-Harandi, A. (2012). Coupled and tripled fixed point theory in partially ordered metric spaces with application to initial value problem. *Mathematical and Computer Modeling,* **57**, 2343-2348.

Aniki, S. A. & Rauf, K. (2019). Stability results on coupled fixed point iterative procedures in complete metric spaces. *Islamic University Multidisciplinary Journal,* **6**(3), 175-186. ISSN 2617-6513.

Aniki, S. A. & Rauf, K. (2021). Stability theorem and results for quadrupled fixed point of contractive type single valued operators. *Iranian Journal of Optimization,* **12**(2).

Abbas, M., Aydi H. & Karapinar E. (2011). Tripled fixed point of multivalued nonlinear contraction mappings in partially ordered metric spaces. *Abstract and Applied Analysis,* (2), 1-12.

Bhaskar, T. G. & Lakshmikantham, V. (2006). Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Analysis,* **65**(7), 1379-1393.

Berinde, V. & Borcut, M. (2011). Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. *Nonlinear Analysis,* **74**(15), 4889-4897.

Ciri´c, L.B. & Lakshmikantham, V. (2009). Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Stochastic Analysis and Applications,* **27**(6), 1246–1259.

Choudhury, B. S. & Kundu, A. A. (2010) Coupled coincidence point result in partially ordered metric spaces for compatible mappings. *Nonlinear Analysis,* **73**(8), 2524-2531.

Karapinar, E. (2010). Coupled foxed point theorems for nonlinear contractions in cone metric spaces. *Computer & Mathematics with Applications,* **59**(12), 3656-3668.

Lakshmikantham, V. & Ciri´c, L. B. (2009). Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Nonlinear Analysis,* **70**(12), 4341-4349.

Rauf, K. & Aniki, S. A. (2021). Quadrupled fixed point theorems for contractive type mappings in partially ordered cauchy spaces. *Confluence Journal of Pure and Applied Sciences,* **4**(1), 18-30.

Rao, K. P. R. & Kishore, G. N. V. (2011). A Unique Common tripled fixed point theorem in partially ordered cone metric spaces. *Bulletin of Mathematical Analysis and Applications,* **3**(4), 213-222.

Sabetghadam, F., Masiha, H. P. & Sanatpour, A. H. (2009). Some coupled fixed point theorems in cone metric spaces. *Fixed Point Theory and Applications,* Article ID 125426. doi:10.1155/2009/125426

Timis, I. (2014). Stability of tripled fixed point iteration procedures for mixed monotone mappings. *Carpathian Mathematical Publications,* **6**(2), 377-388.

FIFTH ORDER BOUNDARY VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS VIA HYBRID FINITE DIFFERENCE BLOCK METHODS

[1.] Soladoye S.O and [2] Yahaya Y.A
[1.]Department of Mathematics and Statistics, Kaduna Polytechnic Kaduna, Nigeria
[2.]Department of Mathematics and Computer Science, Federal University of Technology Minna, Nigeria.
[1.]Corresponding Authur: sikirusoladoye@yahoo.com, Tel. +2348037850404

Abstract

We proposed some symmetric hybrid finite difference methods for the solution of Boundary Value Problems of fifth order Ordinary Differential Equations. The three members block schemes of the Central, Forward and Backward hybrid finite difference methods derived were used simultaneously to obtain the solution of Boundary Value Problems. Two numerical experiments were used to demonstrate the efficiency of the proposed methods.

Keywords: Symmetric, Hybrid, finite difference method, continuous coefficients, Boundary Value Problems.

Introduction

Recently some scholars have proposed some methods of numerical solution of fifth order Boundary value problems of ordinary differential equations. Among whom are

Caglar *et al* (1999) considered the numerical solution of fifth-order linear and non-linear Boundary Value Problems with two-point boundary conditions using sixth-degree B-spline approximation.to construct the method.

Wazwaz (2001) used Adomian decomposition method and the modified form of the method for the numerical solution of fifth-order boundary value problems with two-point boundary conditions.

Hassan and Erturk (2009) used differential transformation method for the numerical solution of fifth-order boundary value problems.

Juan (2009) used variational iteration method for the numerical solution of fifth-order boundary value problems.

Mamadu and Njose (2016) apply Mamadu-Njose polynomials as trial functions in the numerical solution of fifth-order boundary value problems.

Pandey (2017) presented a finite difference method for the solution of fifth order Boundary Value Problems. He transformed the fifth order differential problem into system of differential equations of lower order namely one and four. His method was based on interpolation at the grid points.

Methodology

We consider an approximation of the form:

$$y(x_m) = \sum_{j=0}^{8} a_j Q_j(x_m), \qquad x_{i-2} \le x \le x_{i+2}$$

$$k = 8, \quad m = i-2, i-\frac{3}{2}, i-1, i-\frac{1}{2}, i, i+\frac{1}{2}, i+1, i+\frac{3}{2}, i+2 \tag{2.1}$$

where $Q_j(x)$ are canonical polynomials which are used as the basis functions in the approximation and a_j are coefficients to be determined. To generate$Q_j(x)$, we write equation (1.4) and (1.5) in the form

$$y^n(x) + y(x) = y(x) + f(x, y, y', \dots, y^{n-1}) \tag{2.2}$$

And define a differential operator

$$L^* = \frac{d^3}{dx^3} + 1 \tag{2.3}$$

Then define the canonical polynomial $Q_j(x)$ by

$$L^* Q(x) = x^j, \qquad j = 0,1, k+1 \tag{2.4}$$

We generate the canonical polynomials $Q_j(x)$ by starting with the generating polynomial

$$L^* x^j = j(j-1)(j-2)j^{j-3} + x^j (2.5)$$

Using equations (2.4) and (2.5) it gives

$$L^* x^j = L^*\{j(j-1)(j-2)Q_{j-3}(x) + Q_j\} \tag{2.6}$$

Assume L^{*-1}exists, then we have

$$Q_j(x) = x^j - j(j-1)(j-2)Q_{j-3}(x), \qquad j = 0,1, k+1 \tag{2.7}$$

Equation (2.7) is the recursive relation for generating the canonical polynomials. See Yahaya (1995). Thus, from equation (2.7), we obtain recursively for $j = 0,1,2,\dots,8$ and substituting the result in equation (2.1), we have

$$y_8(x) = a_0 + a_1 x + a_2 x^2 + a_3(x^3 - 6) + a_4(x^4 - 24x) + a_5(x^5 - 60x^2)$$
$$+ a_6(x^6 - 120x^3 + 720) + a_7(x^7 - 420x^4 + 340x) + a_8(x^8 - 240x^5 + 120x^2) \tag{2.8}$$

Where a_j's, $j = 0,1,2, \dots, 8$ are the parameters to be determined. We interpolate at $x = x_m$, where $m = i-2, i-\frac{3}{2}, i-1, i-\frac{1}{2}, i, i+\frac{1}{2}, i+1, i+\frac{3}{2}, i+2$, which gives the system of non-linear equations of the form:

$$y(x_m) = \sum_{j=0}^{8} a_j Q_j(x_m), \tag{2.9}$$

Evaluating the values of $a_0, a_1, a_2, \dots, a_8$ into equation (2.8) and substituting them in equation (2.9) then simplifying, the resulting equation will be of the form:

$$y_8(x) = A(x)y_{i-2} + B(x)y_{i-\frac{3}{2}} + C(x)y_{i-1} + D(x)y_{i-\frac{1}{2}} + E(x)y_i + F(x)y_{i+\frac{1}{2}}$$

$$+G(x)y_{i+1} + H(x)y_{i+\frac{3}{2}} + I(x)y_{i+2} \tag{2.10}$$

where $A(x), B(x), \dots, I(x)$, are known functions to be determined.

We take the first, second…, fifth derivative of equation (2.10) and interpolating each at $x = x_i$, yield the first, second, third, fourth and fifth order derivatives central difference schemes of the form:

$$y'_8(x_i) = \frac{1}{840h}\Big(6y_{i-2} - 64y_{i-\frac{3}{2}} + 336y_{i-1} - 1344y_{i-\frac{1}{2}} + 1344y_{i+\frac{1}{2}} - 336y_{i+1}$$

$$+64y_{i+\frac{3}{2}} - 6y_{i+2}\Big)$$

$$y''_8(x_i) = \frac{1}{2520h^2}\Big(-18y_{i-2} + 256y_{i-\frac{3}{2}} - 2016y_{i-1} + 16128y_{i-\frac{1}{2}} - 28700y_i$$

$$+16128y_{i+\frac{1}{2}} - 2016y_{i+1} + 256y_{i+\frac{3}{2}} - 18y_{i+2}\Big)$$

$$y'''_8(x_i) = \frac{1}{30h^3}\Big(-7y_{i-2} + 72y_{i-\frac{3}{2}} - 338y_{i-1} + 488y_{i-\frac{1}{2}} - 488y_{i+\frac{1}{2}} + 338y_{i+1}$$

$$-72y_{i+\frac{3}{2}} + 7y_{i+2}\Big)$$

$$y^{iv}_8(x_i) = \frac{1}{15h^4}\Big(7y_{i-2} - 96y_{i-\frac{3}{2}} + 676y_{i-1} - 1952y_{i-\frac{1}{2}} + 2730y_i - 1950y_{i+\frac{1}{2}}$$

$$+676y_{i+1} - 96y_{i+\frac{3}{2}} + 7y_{i+2}\Big)$$

$$y^{v}_8(x_i) = \frac{1}{3h^5}\Big(16y_{i-2} - 144y_{i-\frac{3}{2}} + 416y_{i-1} - 464y_{i-\frac{1}{2}} + 464y_{i+\frac{1}{2}} - 416y_{i+1}$$

$$+144y_{i+\frac{3}{2}} - 16y_{i+2}\Big)$$

$$\tag{2.11}$$

Equation (2.11) is of order $[8,8,6,6,4]^T$ with error constants

$$\left[\frac{1}{161280}, \frac{1}{806400}, \frac{-41}{193536}, \frac{-41}{483840}, \frac{13}{2304}\right]^T$$

Also evaluating the first, second…, fifth derivative of equation (2.10) and interpolating each at $x = x_{i-2}$, and replacing $i = i + 2$ yield the first, second, third, fourth and fifth order derivatives forward difference schemes of the form:

$$y'_8(x_i) = \frac{1}{420h}\Big(-2283y_i + 6720y_{i+\frac{1}{2}} - 11760y_{i+1} + 15680y_{i+\frac{3}{2}} - 14700y_{i+2} + 9408y_{i+\frac{5}{2}} - 3920y_{i+3} + 960y_{i+\frac{7}{2}} - 105y_{i+4}\Big)$$

$$y''_8(x_i) = \frac{1}{1260h^2}\Big(29531y_i - 138528y_{i+\frac{1}{2}} + 312984y_{i+1} - 448672y_{i+\frac{3}{2}} + 435330y_{i+2} - 284256y_{i+\frac{5}{2}} + 120008y_{i+3} - 29664y_{i+\frac{7}{2}} + 3267y_{i+4}\Big)$$

$$y'''_8(x_i) = \frac{1}{30h^3}\Big(-2403y_i + 13960y_{i+\frac{1}{2}} - 36706y_{i+1} + 57384y_{i+\frac{3}{2}} - 58280y_{i+2} + 39128y_{i+\frac{5}{2}} - 16830y_{i+3} + 4216y_{i+\frac{7}{2}} - 469y_{i+4}\Big)$$

$$y^{iv}_8(x_i) = \frac{1}{15h^4}\Big(3207y_i - 21056y_{i+\frac{1}{2}} + 61156y_{i+1} - 102912y_{i+\frac{3}{2}} + 109930y_{i+2} - 76352y_{i+\frac{5}{2}} + 33636y_{i+3} - 8576y_{i+\frac{7}{2}} + 967y_{i+4}\Big)$$

$$y^{v}_8(x_i) = \frac{1}{3h^5}\Big(-1296y_i + 9200y_{i+\frac{1}{2}} - 28640y_{i+1} + 51120y_{i+\frac{3}{2}} - 57280y_{i+2} + 41296y_{i+\frac{5}{2}} - 18720y_{i+3} + 4880y_{i+\frac{7}{2}} - 560y_{i+4}\Big) \quad (2.12)$$

Equation (2.12) is of order $[8,7,6,5,4]^T$ with error constants

$\left[\frac{1}{2304}, \frac{-761}{161280}, \frac{29531}{967680}\frac{-267}{1920}, \frac{1067}{2308}\right]^T$ respectively.

Also evaluating the first, second…, fifth derivative equation of (2.10) and interpolating each at $x = x_{i+2}$, and choosing $i = i - 2$ yield the first, second, third, fourth and fifth order derivatives backward difference schemes of the form:

$$y'_8(x_i) = \frac{1}{420h}\Big(105y_{i-4} - 960y_{i-\frac{7}{2}} + 3920y_{i-3} - 9408y_{i-\frac{5}{2}} + 14700y_{i-2} - 15680y_{i-\frac{3}{2}} + 11760y_{i-1} - 6720y_{i-\frac{1}{2}} + 2283y_i\Big)$$

$$y''_8(x_i) = \frac{1}{1260h^2}\Big(3267y_{i-4} - 29664y_{i-\frac{7}{2}} + 120008y_{i-3} - 284256y_{i-\frac{5}{2}} + 435330y_{i-2}$$

$$-448672y_{i-\frac{3}{2}} + 312984y_{i-1} - 138528y_{i-\frac{1}{2}} + 29531y_i\Big)$$

$$y'''_8(x_i) = \frac{1}{30h^3}\Big(469y_{i-4} - 4216y_{i-\frac{7}{2}} + 16830y_{i-3} - 39128y_{i-\frac{5}{2}} + 58280y_{i-2}$$

$$-57384y_{i-\frac{3}{2}} + 36706y_{i-1} - 13960y_{i-\frac{1}{2}} + 2403y_i\Big)$$

$$y^{iv}{}_8(x_i) = \frac{1}{15h^4}\Big(967y_{i-4} - 8576y_{i-\frac{7}{2}} + 33636y_{i-3} - 76352y_{i-\frac{5}{2}} + 109930y_{i-2} - 102912y_{i-\frac{3}{2}}$$

$$+61156y_{i-1} - 21056y_{i-\frac{1}{2}} + 3207y_i\Big)$$

$$y^{v}{}_8(x_i) = \frac{1}{3h^5}\Big(560y_{i-4} - 4880y_{i-\frac{7}{2}} + 18720y_{i-3} - 41296y_{i-\frac{5}{2}} + 57280y_{i-2}$$

$$-51120y_{i-2} + 28640y_{i-1} - 9200y_{i-\frac{1}{2}} + 1296y_i\Big)$$

(2.13)

Equation (2.13) is of order $[8,7,6,5,4]^T$ with error constants

$\left[\frac{1}{2304}, \frac{-761}{161280}, \frac{29531}{967680}, \frac{-267}{1920}, \frac{1067}{2308}\right]^T$ respectively.

Implementation Strategies

Our strategy for implementing the method is such that given an nth order Boundary value problem of ordinary differential equation, we replace the derivatives terms in the equation with an equivalent finite central difference schemes and write the difference equation where the function value is not known. In addition, we replace the derivative Boundary conditions with an equivalent finite forward difference schemes and an equivalent finite backward difference schemes, satisfying the given Boundary conditions. And finally we replace the proposed methods. derivatives terms in the equation with an equivalent finite forward difference schemes and an equivalent finite backward difference schemes, and write the difference equation where the function value is not known, until the number of our unknown functions equals the number of equations. The resulting linear system of simultaneous equations are then solved using MAPLE 17, a mathematical software.

Numerical Examples

The following problems were used to demonstrate the efficiency and accuracy of the

Example 1.

The linear fifth order Boundary value problem

$y^{(v)}(x) = y(x) - 15e^x - 10xe^x, \qquad 0 \le x \le 1$

$y(0) = 0, y'(0) = 1, y''(0) = 0, y(1) = 0, y'(1) = -e$

Theoretical solution: $y(x) = x(1-x)e^x$ Source: Mamadu and Njoseh (2016)

Example 2.

The linear fifth order Boundary value problem

$y^{v}(x) = -4y'(x), \quad 0 \le x \le 1$

$y(0) = 0, y(1) = e\sin 1, y'(0) = 1, y'(1) = e(\sin 1 + \cos 1), y''(0) = 2$

Theoretical solution: $y(x) = e^x \sin x$ Source: Pandey (2017)

Table 1: Approximate solution and absolute error of problem 1

x	y-exact	SFDM	HFDBM	Error in SFDM	Error in HFDBM
0	0.0	0.0	0.0	0.0	0.0
0.1	0.0994653826	0.0994653973	0.0994653833	1.47 E(-8)	6.80 E(-10)
0.2	0.1954244413	0.1954245177	0.1954244439	7.64 E(-8)	2.60 E(-9)
0.3	0.2834703496	0.2834704870	0.2834703529	1.37 E(-7)	3.30 E(-9)
0.4	0.3580379274	0.3580380381	0.3580379290	1.11 E(-7)	1.60 E(-9)
0.5	0.4121803177	0.4121802736	0.4121803155	4.41 E(-8)	2.20 E(-9)
0.6	0.4373085121	0.4373082332	0.4373085056	2.79 E(-7)	6.50 E(-9)
0.7	0.4228880686	0.4228875772	0.4228880592	4.91 E(-7)	9.40 E(-9)
0.8	0.3560865486	0.3560860250	0.3560865396	5.24 E(-7)	9.00 E(-9)
0.9	0.2213642800	0.2213640099	0.2213642753	2.70 E(-7)	4.70 E(-9)
1.0	0.0	0.0	0.0	0.0	0.0

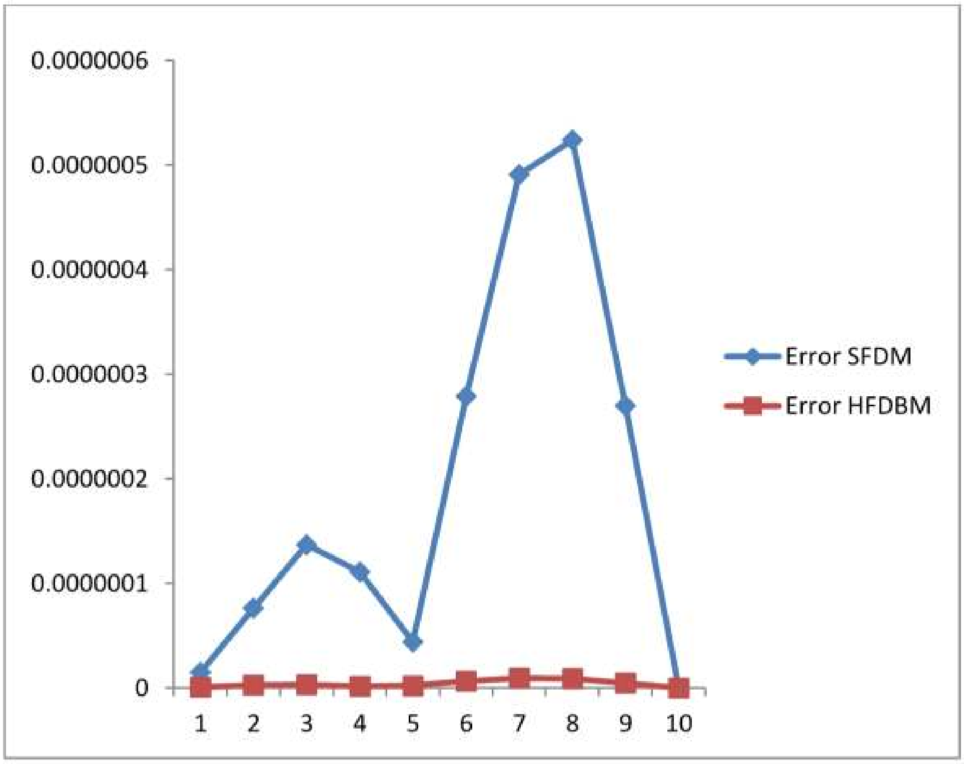


Figure 1: Error graph of Example 1.

Table 2: Approximate solution and absolute error of problem 2

x	y-exact	SFDM	HFDBM	Error in SFDM	Error in HFDBM
0	0.0	0.0	0.0	0.0	0.0
0.1	0.1103329887	0.1103329840	0.1103329888	4.70 E(-9)	1.00 E(-10)
0.2	0.2426552686	0.2426552437	0.2426552694	2.49 E(-8)	8.00 E(-10)
0.3	0.3989105538	0.3989105080	0.3989105561	4.58 E(-8)	2.30 E(-9)
0.4	0.5809439008	0.5809438607	0.5809439052	4.01 E(-8)	4.40 E(-9)
0.5	0.7904390832	0.7904390881	0.7904390899	4.90 E(-9)	6.70 E(-9)
0.6	1.0288456660	1.0288457420	1.0288456750	7.60 E(-8)	9.00 E(-9)
0.7	1.2972951120	1.2972952530	1.2972951200	1.41 E(-7)	8.00 E(-9)
0.8	1.5965053410	1.5965054930	1.5965053470	1.52 E(-7)	6.00 E(-9)
0.9	1.9266733040	1.9266733830	1.9266733070	7.90 E(-8)	3.00 E(-9)
1.0	2.2873552870	2.2873552870	2.2873552870	0.0	0.0

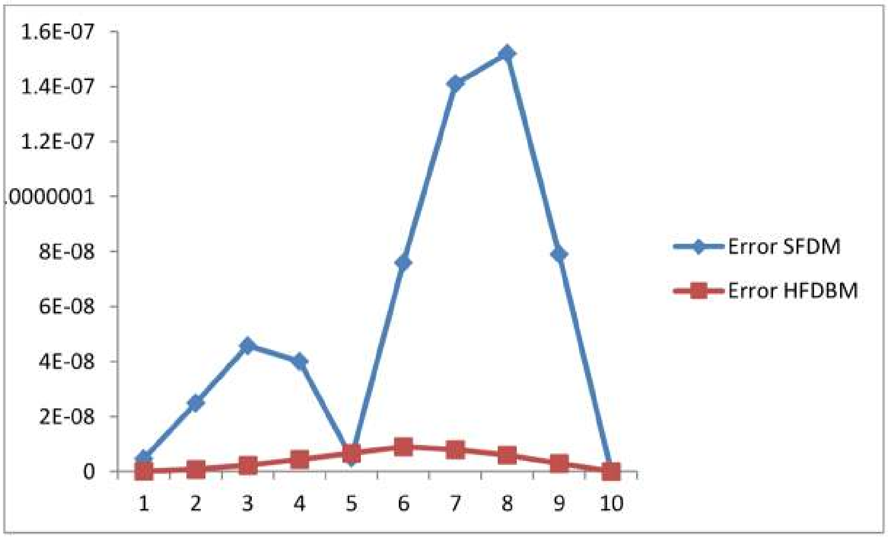


Figure 2: Error graph of Example 2.

SFDM: Standard finite Difference Method

HFDBM: Hybrid Finite Difference Block Method

Table 1: Approximate solution and absolute error of problem 1compared with other existing methods

x	y-exact	Error in HFDBM	Error in six-degree B-spline method	Error in variational iteration method	Error in Mamadu-Njoseh polynomials
0	0.0	0.0	0.0	0.0	0.0
0.1	0.0994653826	6.80 E(-10)	8.00 E(-3)	0.188 E(-4)	1.8107 E(-4)
0.2	0.1954244413	2.60 E(-9)	1.20 E(-3)	1.077 E(-4)	1.1726 E(-3)
0.3	0.2834703496	3.30 E(-9)	5.00 E(-3)	2.477 E(-4)	3.1052 E(-3)
0.4	0.3580379274	1.60 E(-9)	3.00 E(-3)	3.729 E(-4)	5.5441 E(-3)
0.5	0.4121803177	2.20 E(-9)	8.00 E(-3)	4.202 E(-4)	7.7123 E(-3)
0.6	0.4373085121	6.50 E(-9)	6.00 E(-3)	3.643 E(-4)	8.7510 E(-3)
0.7	0.4228880686	9.40 E(-9)	0.0000	2.364 E(-4)	8.0231 E(-3)
0.8	0.3560865486	9.00 E(-9)	9.00 E(-3)	1.158 E(-4)	5.4655 E(-3)
0.9	0.2213642800	4.70 E(-9)	9.00 E(-3)	0.876 E(-4)	1.9985 E(-3)
1.0	0.0	0.0	0.0	0.0	0.0

Conclusion

The numerical experiments in this paper shows that the results from HFDBM are consistent and convergent to the theoretical solution and also compete favourably with standard finite difference method. (see figure 1 and 2). The newly derived method displayed its superiority over those of six-degree spline method in Caglar *et al* (1999), variational iterational method in Juan (2009) and Mamadu-Njoseh polynomials inMamadu-Njoseh (2016).

References

Abdul-Majid Wazwaz (2001). The Numerical Solution of Fifth-order boundary value problems by the Decomposition Method. *Journal of the Computational and Applied Mathematics*. 136 (2001) 259-270. www.elsevier.com/locate/cam

Caglar H. N., Caglar S. H. and Twizell E. E. (1999). The Numerical Solution of Fifth Order Boundary Value Problems with Sixth Degree B-Spline Functions. Applied Mathematics Letters. 12 (1999) 25-30

Hassan Abdel-Halim I. H. and Erturk Vedat Suat (2009). Solution of Different Types of the Linear and Non-linear Higher Order Boundary Value Problems by the differential Transformation Method. *European Journal of Pure and Applied Mathematics*. Vol. 2, No. 3, 2009 (426-447) ISSN 1307-5543 – www.ejpam.com

Ignatius, N. Njose and Ebimene, J. Mamadu (2016). Numerical Solution of Fifth Order Boundary Value Problems using Mamadu-Njose Polynomials. Science World Journal. Vol. 11. (No 4) 2016. www.scienceworldjournal.org ISSN 1597-6343.

Juan Zhang (2009). Numerical Solution of fifth order Boundary value problems by the Variational Iteration Method. Computer and Mathematics with Applications Journal homepage:www.elsevier.com/locate/comwa 58(2009) 2347-2350.

Pandey, P. K. (2017). A Numerical Method for the Solution of fifth order Boundary Value Problem in Ordinary Differential Equation. *Vladikakazian Mathematical Journal*. Volume 19, 50-57.

Yahaya Y. A. (1995). An Off-grid Finite Difference Method for Two –point Boundary-value Problems. MSc. Thesis (Unpublished) University of Jos, Nigeria.